ON CONFORMAL MAPPING OF REGIONS BOUNDED BY SMOOTH CURVES

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1. Introduction. The object of this note is the proof of the following theorem.

THEOREM. *Suppose C is a simple closed curve which contains the origin of its interior R and which satisfies the following hypotheses:*

(a) *C possesses a continuously turning tangent and the tangent angle $\alpha(s)$ considered as a function of the arc length s has the modulus of continuity $\beta(t)$, that is,*

$$|\alpha(s \pm t) - \alpha(s)| \leqq \beta(t), \qquad t > 0, \tag{1}$$

where $\beta(t)$ is a nondecreasing function of t and $\lim_{t\to 0+} \beta(t) = 0$.

(b) *There exists a constant k such that if P_1 and P_2 are two points of C and Δs is the (shorter) arc between them, then*

$$\frac{\Delta s}{\overline{P_1P_2}} \leqq k. \tag{2}$$

(c) *The diameter of C does not exceed D, and the distance of the origin from C is at least equal to σ, $\sigma > 0$.*

Suppose that $w = f(z)$ maps the circle $|z| < 1$ conformally onto R such that $f(0) = 0$. Then, for every $p > 0$, there exists a constant A_p which depends only on p, the constants k, D, σ, and the function $\beta(t)$—and in no other way on the curve C—such that uniformly for $0 \leqq \rho < 1$

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |f'(\rho e^{i\theta})|^{\pm p} d\theta\right\}^{1/p} \leqq A_p. \tag{3}$$

An explicit expression in terms of these parameters is obtained for A_p.

The fact that the integral in (3) remains bounded for $0 \leqq \rho < 1$ under the assumption that C has continuously turning tangents was proved in an earlier paper of the writer [4, p. 362].[1] The emphasis in the present note is upon the fact that the constant A_p *depends only on the parameters indicated and is expressed explicitly in terms of these quantities.* This result is of use when an estimate for the integral in (3) is desired which holds uniformly for the mapping functions of a *family* of curves C.

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[1] Numbers in the brackets refer to the bibliography at the end of the paper.

Before proving the theorem we state an application. By a theorem of F. Riesz, $f'(\rho e^{i\theta})$ has limit values as $\rho \to 1$ for almost all θ and (3) holds for $\rho = 1$. By use of Hölder's inequality we have for $p > 1$, $z = e^{i\theta}$, $z_0 = e^{i\theta_0}$:

$$\begin{aligned} |f(z) - f(z_0)| &= \left| \int_{\theta_0}^{\theta} f'(e^{it}) ie^{it} dt \right| \\ &\leqq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f'(e^{it})|^p dt \right\}^{1/p} |\theta - \theta_0|^{(p-1)/p} (2\pi)^{1/p} \\ &\leqq A_p (2\pi)^{1/p} |\theta - \theta_0|^{1-1/p}. \end{aligned}$$

From this it is easily seen that for $|z| = |z_0| = 1$

$$|f(z) - f(z_0)| \leqq 2^{2/p-1} A_p \pi |z - z_0|^{1-1/p}, \tag{4}$$

and by a theorem in [6, p. 669],[2] this inequality holds also for $|z| < 1$.

Let $\phi(w)$ denote the inverse function of $f(z)$. Then if w_0, w are points of C and Δs is the length of the (shorter) arc between w_0 and w, then (note that $\phi(w)$ is an absolutely continuous function along C)

$$\begin{aligned} |\phi(w) - \phi(w_0)| &= \left| \int_{w_0}^{w} {}^{(C)} \phi'(u) du \right| \\ &\leqq \left\{ \int_C |\phi'(w)|^p |dw| \right\}^{1/p} (\Delta s)^{1-1/p} \\ &\leqq \left\{ \int_0^{2\pi} |f'(e^{i\theta})|^{-p} |f'(e^{i\theta})| d\theta \right\}^{1/p} (\Delta s)^{1-1/p}. \end{aligned}$$

Hence by (2),

$$|\phi(w) - \phi(w_0)| \leqq (2\pi)^{1/p} (A_{p-1})^{(p-1)/p} (k |w - w_0|)^{1-1/p}, \tag{5}$$

and by the theorem just quoted, this is true for all w in R (w_0 on C). Combining (4) and (5) we obtain the following corollary.

COROLLARY. *Under the hypotheses of the theorem there exists for every δ, $0 < \delta < 1$, a constant B which depends only on δ and on k, D, σ, and the function $\beta(t)$ such that for $|z_0| = 1$, $|z| \leqq 1$,*

$$\frac{1}{B} |z - z_0|^{1/(1-\delta)} \leqq |f(z) - f(z_0)| \leqq B |z - z_0|^{1-\delta}.$$

[2] This theorem is as follows: *Let R be a region bounded by a simple closed curve Γ and let $f(z)$ be regular in R and continuous in $R + \Gamma$. If for a point $z_0 \in \Gamma$, and for all $z \in \Gamma$: $|f(z) - f(z_0)| \leqq M |z - z_0|^\alpha$, where $\alpha > 0$, then this holds also for all z in R.*

(*If we set* $\delta = 1/p$, $B = \text{Max}\ (2^{2/p-1}\pi A_p,\ (2\pi)^{1/(p-1)} k A_{p-1})$.)

The existence of a relation of this form at every point z_0 and even uniformly along the circle $|z| = 1$ for a *fixed* curve C is known [5], [1], and was proved in a different way. The point of our corollary lies again in the fact that the dependence of the constant B upon the parameters which characterize C is given.

2. **Lemmas.** We shall need the following lemmas.

LEMMA 1. *Under the hypotheses of the theorem there exists a positive* $\rho < \sigma$ *which depends only on* k, σ, *and the function* $\beta(t)$, *such that any circle of radius less than or equal to* ρ *about any point* P *of* C *intersects* C *in exactly two points and that the length of the subarc of* C *contained in this circle does not exceed* 3ρ.

PROOF. Let η, $0 < \eta < k\sigma$, be so chosen that $\beta(t) \leqq 1/8$ for $0 < t \leqq \eta$. Then we may take $\rho = \eta/k$. To see this we describe a circle K of radius $r \leqq \rho$ about P. Since $r \leqq \rho < \sigma$ there are points of C *exterior* to K, so that K intersects C. Let P_1 be the *first* point of intersection of K and C, which is met when C is traversed from P in *one* direction, and let P_2 be the *first* such point which one meets in going from P along C in the opposite direction. (P_1 and P_2 are distinct, for otherwise the curve C could not have any points in the exterior of K.)

Suppose now there existed a third point of intersection of K and C, say P_3. Assume, without loss of generality, that of the two complementary arcs PP_1P_3 and PP_2P_3 of C, the length Δs of the first does not exceed that of the second. Since by (2)

$$\Delta s \leqq k \cdot \overline{PP_3} = kr \leqq k\rho = \eta,$$

it follows that for all points s of the arc PP_1P_3 (s_0 corresponds to P)

$$|\alpha(s) - \alpha(s_0)| \leqq \beta(\eta) \leqq 1/8.$$

Hence, the arc PP_1P_3 of C lies within the angle which is formed by two straight lines through P each of which forms an angle of opening $1/8$ with the tangent to C at P. Then it follows that one of the angles between the tangent to C at P and the chord P_1P_3 of C is between $\pi/2 - 1/8$ and $\pi/2$. There exists a point $s = s^*$ of the arc PP_1P_3 between P_1 and P_3 such that $\alpha(s^*)$ is equal to the angle of inclination of P_1P_3. Hence $|\alpha(s^*) - \alpha(s_0)| \geqq \pi/2 - 1/8$, which contradicts the inequality stated above.

We prove now the statement concerning the length of the subarc c of C which is contained in the interior of K. Let $z(s) = x(s) + iy(s)$ denote the parametric representation of C in terms of s. Suppose that the interval $s_1 \leqq s \leqq s_2$ corresponds to c; $z_1 = z(s_1)$, $z_2 = z(s_2)$ are the end points of c. Then

$$z_2 - z_1 = \int_{s_1}^{s_2} (x'(s) + iy'(s))ds = (s_2 - s_1)(x'(\xi_1) + iy'(\xi_2))$$

where $s_1 \leqq \xi_1, \xi_2 \leqq s_2$. Hence

$$\begin{aligned} | z_2 - z_1 | &= (s_2 - s_1) | x'(\xi_1) + iy'(\xi_1) + i(y'(\xi_2) - y'(\xi_1) | \\ &\geqq (s_2 - s_1)\{ | x'(\xi_1) + iy'(\xi_1) | - | y'(\xi_2) - y'(\xi_1) | \}. \end{aligned}$$

Since $x'(s) = \cos \alpha(s)$, $y'(s) = \sin \alpha(s)$ and $|\xi_1 - s_0| \leqq \eta$, $|\xi_2 - s_0| \leqq \eta$,

$$\begin{aligned} | z_2 - z_1 | &\geqq (s_2 - s_1)\{1 - | y'(\xi_2) - y'(\xi_1) | \} \\ &\geqq (s_2 - s_1)(1 - 2\beta(\eta)) \geqq 3(s_2 - s_1)/4, \end{aligned}$$

or

$$s_2 - s_1 \leqq \frac{4}{3} | z_2 - z_1 | \leqq \frac{4}{3} 2\rho < 3\rho.$$

LEMMA 2 (MODULUS OF CONTINUITY). *Suppose that the hypotheses of the theorem are satisfied. Let* $r_0 = \exp [-\pi^2 D^2/2\rho^2]$, *where* ρ *is the number given in Lemma* 1. *Then for any two points* z, z_0 *of* $|z| = 1$ *for which* $|z - z_0| \leqq r \leqq r_0$:

$$| f(z) - f(z_0) | \leqq \frac{\pi D}{(2 \log (1/r))^{1/2}}.$$

PROOF. Let z_0 be a point of $|z| = 1$ and h_r the part of the circle $|z - z_0| = r$ which is contained in $|z| < 1$. Then by a theorem of J. Wolff [7, p. 217], there exists for every r, $0 < r < 1$, an r_1, $r < r_1 < r^{1/2}$, such that the image of h_{r_1} by means of $w = f(z)$ is a cross-cut γ_{r_1} of R whose length $l_{r_1} \leqq (2\pi A/\log (1/r))^{1/2}$, where A is the area of R. Since $A \leqq \pi D^2/4$, we have

$$l_{r_1} \leqq \frac{\pi D}{(2 \log (1/r))^{1/2}}. \tag{6}$$

Assume now $r \leqq r_0$. Then the region $\{ |z - z_0| < r_1, |z| < 1 \}$ is mapped onto a subregion of R which is bounded by γ_{r_1} and an arc of C. If P_1 and P_2 are the end points of γ_{r_1}, then

$$\overline{P_1 P_2} \leqq \frac{\pi D}{(2 \log (1/r))^{1/2}} \leqq \rho < \sigma.$$

Hence, by Lemma 1, one of the two arcs of C between P_1 and P_2, say

c_{r_1}, is contained in the circle of radius $\pi D/(2 \log (1/r))^{1/2}$ about P_1. Because of (6), this circle contains γ_{r_1} and hence also the region bounded by γ_{r_1} and c_{r_1}. Since $\rho<\sigma$, the origin is in the exterior of this circle, and it follows that c_{r_1} is the image of the arc $\{|z|=1, |z-z_0|\leqq r_1\}$. Thus, if $z=e^{i\theta}$ and $|z-z_0|\leqq r$, then

$$|f(z)-f(z_0)| \leqq \frac{\pi D}{(2 \log (1/r))^{1/2}}.$$

LEMMA 3 (PROPERTIES OF AN AUXILIARY FUNCTION). *Suppose* $0=\theta_1<\theta_2< \cdots <\theta_n <\theta_{n+1}=2\pi$ *and* $\tau_1, \tau_2, \cdots, \tau_n, \tau_{n+1}=\tau_1+2\pi$ *are two sets of real numbers. Let* $\beta_m=(\tau_m-\tau_{m+1})/\pi$, $m=1, 2, \cdots, n$, *and for* $|z|<1$

$$g(z) = \prod_{m=1}^{n} (1 - e^{-i\theta_m}z)^{\beta_m};$$

each factor is single-valued and analytic for $|z|<1$ *if that branch of the power function is chosen which reduces to* 1 *for* $z=0$. *If this branch of* $\arg g(z)$ *is taken, then*

$$\lim_{z\to e^{i\theta}} \arg g(z) = \arg g(e^{i\theta}) = \tau_m - \theta - \pi/2 - \omega, \quad \textit{when} \quad \theta_m < \theta <\theta_{m+1},$$

where

$$\omega = \frac{1}{2\pi}\sum_{m=1}^{n} \tau_m(\theta_{m+1} - \theta_m) - \frac{3}{2}\pi.$$

Furthermore $\arg g(z)$ *is bounded in* $|z|<1$.

PROOF. We define the function $\tau(\theta)$ by the relations $\tau(\theta)=\tau_m$ for $\theta_m\leqq\theta<\theta_{m+1}$, $m=1, 2, \cdots, n$, $\tau(\theta+2\pi)=\tau(\theta)+2\pi$. Then we may write[3] for $|z|<1$:

$$\begin{aligned}
\log g(z) &= \frac{1}{\pi}\sum_{m=1}^{n} (\tau_m - \tau_{m+1}) \log (1 - e^{-i\theta_m}z) \\
&= -\frac{1}{\pi}\int_{\theta=0}^{2\pi} \log (1 - e^{-i\theta}z)d\tau(\theta) \\
&= -\frac{1}{\pi}\int_{\theta=0}^{2\pi} \log (1 - e^{-i\theta}z)d[\tau(\theta) - \theta] \\
&\quad - \frac{1}{\pi}\int_{0}^{2\pi} \log (1 - e^{-i\theta}z)d\theta.
\end{aligned}$$

[3] Cf., for this representation of $\log g(z)$ in form of a Stieltjes integral, E. Study [3].

Now $\int_0^{2\pi} \log (1-e^{-i\theta}z)d\theta=0$. Upon integration by parts we find

$$\log g(z) = -\frac{1}{\pi}\Big[\log (1 - e^{-i\theta}z)(\tau(\theta) - \theta)\Big]_0^{2\pi} + \frac{iz}{\pi}\int_0^{2\pi} \frac{\tau(\theta) - \theta}{e^{i\theta} - z} d\theta$$

and since $\tau(\theta)-\theta$ has the period 2π, the integrated part vanishes. Since

$$\frac{z}{e^{i\theta} - z} = \frac{z}{e^{i\theta} - z} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}\,\frac{e^{i\theta} + z}{e^{i\theta} - z} - \frac{1}{2},$$

we finally obtain

$$\text{(7)}\quad \log g(z) = \frac{i}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}(\tau(\theta) - \theta)d\theta - \frac{i}{2\pi}\int_0^{2\pi} (\tau(\theta) - \theta)d\theta.$$

The last integral

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} (\tau(\theta) - \theta)d\theta &= \frac{1}{2\pi}\sum_{m=1}^{n}\int_{\theta_m}^{\theta_{m+1}} \tau(\theta)d\theta - \pi\\ &= \frac{1}{2\pi}\sum_{m=1}^{n} \tau_m(\theta_{m+1} - \theta_m) - \pi.\end{aligned}$$

The conclusion of the lemma follows easily from the representation (7) if we take the imaginary parts of both sides.

3. **Proof of the theorem.** (i) The correspondence between C and the unit circle $|z|=1$ is given by $w=f(e^{i\theta})$, $0\leqq\theta\leqq 2\pi$. Let $s(\theta)$ denote the variable arc length of C expressed as a function of θ. By Lemma 2, the subarc c of C given by $w=f(e^{i\theta})$, $\theta_1\leqq\theta\leqq\theta_2$, where $\theta_2-\theta_1\leqq r\leqq r_0$, lies within the circle of radius $(\pi D/(2\log(1/r))^{1/2}<\rho$ about the point $w_1=f(e^{i\theta_1})$. By Lemma 1 the length of c is $\leqq 3\rho<3\sigma$. Since the total length of C is at least $2\pi\sigma$, it follows that c is the *shorter* arc of C between its end points, and thus by (2)

$$0 \leqq s(\theta_2) - s(\theta_1) \leqq k\,|f(e^{i\theta_2}) - f(e^{i\theta_1})| \leqq \frac{k\pi D}{(2\log(1/r))^{1/2}}\,.$$

Let $\tau(\theta)=\alpha[s(\theta)]$. Given any $\epsilon>0$ there exists a positive $\delta_1\leqq k\rho$ which depends only on the function $\beta(t)$ such that

$$\text{(8)}\qquad \beta(t) \leqq \epsilon \qquad \text{for } 0 < t \leqq \delta_1.$$

Let δ be so chosen that

$$\text{(9)}\qquad \frac{k\pi D}{(2\log(1/r))^{1/2}} \leqq \delta_1 \qquad \text{for } r \leqq \delta.$$

Then for any θ_0, θ, $0 \leqq \theta$, $\theta_0 \leqq 2\pi$, we have by (8), (9), and (1)

$$(10) \qquad |\tau(\theta) - \tau(\theta_0)| \leqq \epsilon \qquad \text{if } |\theta - \theta_0| \leqq \delta.$$

Clearly δ depends only on ϵ and D, σ, k, and the function $\beta(t)$, as $\delta = \exp[-k^2\pi^2 D^2/2\delta_1^2]$ where $\beta(\delta_1) \leqq \epsilon$, $\delta_1 \leqq k\rho$.

(ii) Let $n = [2\pi/\delta] + 1$, so that $2\pi/n < \delta$. Let $\theta_k = (k-1)2\pi/n$, $k = 1, 2, \cdots, n+1$, and $\tau_k = \tau(\theta_k)$. We have then $\tau_{n+1} = \tau_1 + 2\pi$. With these two sets of numbers θ_k, τ_k we form the function $g(z)$ of Lemma 3. Consider the quotient $F(z) = (f'(z)/g(z))e^{-i\omega}$, $|z| < 1$, where ω is the constant in Lemma 3. The $\log F(z) = \log f'(z) - \log g(z) - i\omega$ is single-valued and analytic for $|z| < 1$, if the same branch of $\log g(z)$ is chosen as in Lemma 3 and if $\log f'(z)$ is so determined that $\log f'(0)$ is real. By a Theorem of Lindelöf [2], $\arg f'(z)$, which is harmonic for $|z| < 1$, is continuous in $|z| \leqq 1$ and

$$\arg f'(e^{i\theta}) = \tau(\theta) - \theta - \pi/2.$$

Hence, in every interval $\theta_k < \theta < \theta_{k+1}$, by (10)

$$|\arg f'(e^{i\theta}) - \arg [g(e^{i\theta})e^{i\omega}]| = |\tau(\theta) - \tau_k| \leqq \epsilon.$$

Furthermore, since $\arg F(z)$ is *bounded* in $|z| < 1$, we have

$$(11) \qquad |\arg f'(z) - \arg [g(z)e^{i\omega}]| \leqq \epsilon \qquad \text{for } |z| < 1.$$

(iii) Given any $p > 0$, choose $\epsilon = 1/3p\kappa$ where $\kappa = 2e/\log 2$. Applying an inequality on conjugate functions[4] [4, p. 356], we obtain from (11): For $0 \leqq \rho < 1$,

$$(12) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{f'(\rho e^{i\theta})e^{-i\omega}}{f'(0)g(\rho e^{i\theta})} \right|^{\pm 2p} d\theta \leqq \frac{e^{2p\epsilon}}{1 - 2p\epsilon\kappa} \leqq \frac{e^{1/\kappa}}{1 - 2/3} = 3e^{1/\kappa}.$$

Furthermore,

$$\begin{aligned} \int_0^{2\pi} |g(\rho e^{i\theta})|^{\pm 2p} d\theta &= \int_0^{2\pi} \prod_{m=1}^{n} |1 - \rho e^{i(\theta - \theta_m)}|^{\pm 2p\beta_m} d\theta \\ &\leqq \int_0^{2\pi} \prod_{m=1}^{n} \left[\frac{4}{|1 - \rho e^{i(\theta - \theta_m)}|} \right]^{2p|\beta_m|} d\theta \\ &\leqq \int_0^{2\pi} \prod_{m=1}^{n} \left| 2 \csc \frac{\theta - \theta_m}{2} \right|^{2p|\beta_m|} d\theta. \end{aligned}$$

[4] The theorem referred to is the following: *Suppose $\phi(z) = U(z) + iV(z)$ is regular for $|z| < 1$, suppose $U(0) = 0$ and $|V(z)| \leqq \eta$ for $|z| < 1$. Then there exists an absolute constant κ, which may be taken as $\kappa = 2e/\log 2$, such that for every q, $0 < q < 1/\kappa\eta$,* $(1/2\pi)\int_0^{2\pi} e^{q|\phi(\rho e^{i\theta})|} d\theta \leqq e^{\eta q}/(1 - \eta q\kappa)$, $0 \leqq \rho < 1$. We apply this theorem with $\phi(z)$ $\log F(z) - \log f'(0)$, $\eta = \epsilon$, $q = 2p$.

Since $2p|\beta_m| \leqq 2p\epsilon/\pi < 1/\kappa\pi$, we have

$$\begin{aligned}(13)\qquad &\frac{1}{2\pi}\int_0^{2\pi} |g(\rho e^{i\theta})|^{\pm 2p} d\theta \\ &\qquad \leqq 2^{n/\pi\kappa}\frac{1}{2\pi}\int_0^{2\pi}\prod_{m=1}^{n}\left|\csc\frac{\theta-\theta_m}{2}\right|^{1/\pi\kappa} d\theta = M_n,\end{aligned}$$

where M_n depends only on n, which in turn depends only on p, D, σ, k, and the function $\beta(t)$.

Thus, if $z = \rho e^{i\theta}$, we have

$$\begin{aligned}(14)\qquad &\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(z)}{f'(0)}\right|^{\pm p} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(z)}{f'(0)g(z)}\right|^{\pm p} |g(z)|^{\pm p} d\theta \\ &\leqq \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(z)}{f'(0)g(z)}\right|^{\pm 2p} d\theta\right\}^{1/2}\left\{\frac{1}{2\pi}\int_0^{2\pi}|g(z)|^{\pm 2p} d\theta\right\}^{1/2} \\ &\leqq (3e^{1/\kappa}M_n)^{1/2}\end{aligned}$$

by (12) and (13). Since by Cauchy's inequality $\sigma \leqq |f'(0)| \leqq D$, we have

$$(15)\qquad |f'(0)|^{\pm p} \leqq \operatorname{Max}\left[D^p, \frac{1}{\sigma^p}\right],$$

and the conclusion of the theorem follows from (14) and (15) with $A_p = [3e^{1/\kappa}M_n]^{1/2p}\cdot\operatorname{Max}[D, 1/\sigma]$.

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