APPROXIMATION TO CONJUGATE FUNCTIONS

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The object of the present note is the establishment of the following theorem:

THEOREM 1. *If* $\{T_n(x)\}$ *is a sequence of trigonometric polynomials of order* n, *and if*

(1) $$f(x) - T_n(x) = O(n^{-\alpha}) \qquad (\alpha > 0) \textit{ uniformly in } x,$$

then the conjugate function and the conjugate trigonometric polynomials satisfy

(2) $$\bar{f}(x) - \overline{T}_n(x) = O(n^{-\alpha} \log n) \qquad \textit{uniformly in } x.$$

Furthermore the latter order is in general the best possible.

For the sequence $\{T_n(x)\}$ the sequence of partial sums of the Fourier series of $f(x)$, Salem and Zygmund [3][1] showed that $f(x) - s_n(x) = 0(n^{-\alpha})$ for $\alpha > 0$ uniformly in x implied that $\bar{f}(x) - \bar{s}_n(x) = O(n^{-\alpha})$. Kawata [2] pointed out that for the sequence of Fejer means of the Fourier series of $f(x)$ and for $0 < \alpha < 1$

(3) $$f(x) - \sigma_n(x) = O(n^{-\alpha}) \qquad \text{uniformly in } x$$

implied

(4) $$\bar{f}(x) - \bar{\sigma}_n(x) = O(n^{-\alpha}) \qquad \text{uniformly in } x$$

while (1) for $\alpha = 1$ implied only

(5) $$\bar{f}(x) - \bar{\sigma}_n(x) = O(n^{-1} \log n) \qquad \text{uniformly in } x.$$

For $\alpha > 1$, of course, relation (3) implies that $f(x)$ is a constant.

Suppose first that $0 < \alpha < 1$. If condition (1) is satisfied, then by a theorem of S. Bernstein [1] $f(x) \in \text{Lip } \alpha$. Hence, by another result of Bernstein [1], relation (3) holds and thus (4) follows.

The polynomial $Q_n(x) = T_n(x) - \sigma_n(x)$ has $|Q_n(x)| \leqq Kn^{-\alpha}$ uniformly with respect to x. Hence, there is a constant C such that

(6) $$|\overline{Q}_n(x)| \leqq C_n^{-\alpha} \log n.$$

The combination of (4) and (6) gives (2).

The argument for $\alpha \geqq 1$ proceeds in a similar fashion to that used

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[1] Numbers in brackets refer to the references at the end of the paper.

by Salem and Zygmund [3]. Choose β so that $\alpha=\beta+\epsilon$ and $0<\epsilon<1$. If we let $T_{-1}(x)=0$, $\Delta_n(x)=T_n(x)-T_{n-1}(x)$, then (1) implies $f(x) = \sum_{k=0}^{\infty} \{T_k(x)-T_{k-1}(x)\} = \sum_{k=0}^{\infty} \Delta_k(x)$. Hence, $f(x)-T_n(x) = \sum_{k=n}^{\infty} \Delta_k(x) = O(n^{-\alpha})$. If we now let $g(x)=\sum_{k=1}^{\infty} k^{\beta}\Delta_k(x)$, we have

$$\begin{aligned} g(x) - \sum_{k=1}^{n} k^{\beta}\Delta_k(x) &= \sum_{k=n+1}^{\infty} k^{\beta}\{T_{k+1}(x) - T_k(x)\} \\ &= \{f(x) - T_{n+1}(x)\}(n+1)^{\beta} \\ &\quad + \sum_{k=n+2}^{\infty} \{k^{\beta} - (k-1)^{\beta}\}\{f(x) - T_k(x)\} \\ &= O(n^{-\epsilon}) \qquad \text{uniformly in } x. \end{aligned}$$

Hence by the portion of the theorem already established

$$\bar{g}(x) - \sum_{k=1}^{n} k^{\beta}\bar{\Delta}_k(x) = O(n^{-\epsilon}\log n) \qquad \text{uniformly in } x.$$

Consequently if $S_n(x)=\sum_{j=n}^{\infty} j^{\beta}\Delta_j(x)$,

$$\begin{aligned} \bar{f}(x) - \bar{T}_n(x) &= \sum_{k=n+1}^{\infty} \bar{\Delta}_k(x) = \sum_{k=n+1}^{\infty} k^{-\beta}\{S_k(x) - S_{k+1}(x)\} \\ &= (n+1)^{-\beta}S_{n+1}(x) + \sum_{k=n+2}^{\infty} \{k^{-\beta} - (k-1)^{-\beta}\}S_k(x) \\ &= O(n^{-\alpha}) \qquad \text{uniformly in } x. \end{aligned}$$

In order to justify the final remark of the theorem, we note that $f(x)=0$ and the polynomials $T_n(x)=n^{-\alpha}\sum_{k=1}^{n} k^{-1}\sin kx$ satisfy (1) while at $x=0$, $\bar{T}_n(x)$ is of the exact order $n^{-\alpha}\log n$.

References

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2. T. Kawata, *The Lipschitz condition of a function and Fejer means of Fourier series*, Kodai Mathematical Seminar Reports, 1949, pp. 1–4.

3. R. Salem and A. Zygmund, *The approximation by partial sums of Fourier series*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 14–22.

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