GAMES AND SUB-GAMES[1]

SEYMOUR SHERMAN

The determination of the value and optimal strategies of a zero-sum, two-person game with a finite number of pure strategies can be a lengthy process, involving, among other things, the calculation of the value

$$\sup_x \inf_y (xB, y),$$

where B is a real matrix with m rows and n columns, x ranges over the set of row vectors with m components, all non-negative and adding up to one, y ranges over the corresponding set of n-component column vectors, and the pay-off, (xB, y), indicates the inner product of the two vectors xB and y. One device which may simplify a game computation is that of "dominance" or "majorization" [vNM, p. 174] by which the solution of a game is reduced to the solution of a smaller game, that is, one with a smaller number of pure strategies. There is another device which, when conditions are right, may simplify the solution of a game by reducing it to the solution of smaller games. This device, presented here, gives either the value or a bound for it, depending on the information available about the sub-games. It also gives an optimal strategy or a strategy sufficient to insure an outcome not worse than that predicted by the aforementioned bound. It is particularly effective when there are rows (or columns) in B, which are constant or have large constant segments.

Let B be a game matrix (rows maximizing) decomposed into

$$B = \{B^i_j \mid 1 \leqq i \leqq M, 1 \leqq j \leqq N\},$$

where B^i_j is a sub-matrix with m_i rows and n_j columns (the m_i rows being independent of j and the n_j columns being independent of i). Let the value of B be v and the value of B^i_j be v^i_j. Let the set of optimal strategies for the first player in the game be $X = \{x\}$ and the set of optimal strategies for the first player in the sub-game be $X^i_j = \{x^i\}$. Let Y and Y^i_j represent the corresponding sets for the

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[1] This originally appeared in a RAND report: *Total reconnaissance with total countermeasures: Simplified model*, August 5, 1949, P-106, Rand Corporation, Santa Monica, California. For the definitions in game theory see [vNM]. See the bibliography at the end of the paper.

second players. Let $\overline{B}$ be the $M \times N$ matrix with entries v_j^i. Let $\bar{x} = \{\bar{x}_i \mid 1 \leqq i \leqq M,\ \bar{x}_i \geqq 0,\ \sum \bar{x}_i = 1\}$ be a typical optimal strategy for the first player in the game with matrix $\overline{B}$, $\overline{X}$ the set of optimal strategies for the first player in $\overline{B}$, and $\bar{y}$ and $\overline{Y}$ the analogous items for the second player. Let $\bar{v}$ be the value of the game with matrix $\overline{B}$.

THEOREM. *$\bigcap_j X_j^i \neq \Lambda$ for each i implies $v \geqq \bar{v}$. If $x^i \in \bigcap_j X_j^i$ for each i and $\bar{x} \in \overline{X}$, then by playing the vector $\{\bar{x}_1 x^1, \bar{x}_2 x^2, \cdots, \bar{x}_M x^M\}$ (where by this notation we mean the vector each of whose first m_1 components are $\bar{x}_1$ multiplied by the appropriate one of the m_1 components of x^1, and so on) the first player may assure himself of a pay-off of at least $\bar{v}$.*

PROOF. Let a typical strategy for II in game with matrix B be

$$y = \{\beta^1 \bar{y}_1, \cdots, \beta^N \bar{y}_N\}$$

where $\bar{y}_j$ is a vector with n_j non-negative components adding up to one and $\beta^j \geqq 0$ for each j, $\sum \beta^j = 1$. If I plays

$$\{\bar{x}_1 x^1, \cdots, \bar{x}_M x^M\}$$

then the pay-off

$$\sum_{j=1}^{N} \left(\sum_{i=1}^{M} \bar{x}_i x^i B_j^i \beta^j \bar{y}_j \right) \geqq \sum_{j=1}^{N} \left(\sum_{i=1}^{M} \bar{x}_i v_j^i \beta^j \right) \geqq \bar{v}.$$

COROLLARY. *$\bigcap_i Y_j^i \neq \Lambda$ for each j implies $v \leqq \bar{v}$. If $y_j \in \bigcap_i Y_j^i$ for each j and $\bar{y} \in \overline{Y}$, then by playing the vector $\{\bar{y}^1 y_1, \cdots, \bar{y}^N y_N\}$ the second player may limit his losses to $\bar{v}$.*

COROLLARY.[2] *$\bigcap_j X_j^i \neq \Lambda$ for each i and $\bigcap_i Y_j^i \neq \Lambda$ for each j implies $v = \bar{v}$. $x^i \in \bigcap_j X_j^i$, $y_j \in \bigcap_i Y_j^i$, $\bar{x} \in \overline{X}$ and $\bar{y} \in \overline{Y}$ implies $\{x_1 x^1, \cdots, x_M x^M\} \in X$ and $\{\bar{y}^1 y_1, \cdots, \bar{y}^N y_N\} \in Y$.*

BIBLIOGRAPHY

[vNM] J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, 2d ed., Princeton University Press, 1947.

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[2] Similar results (unpublished) have been obtained by Gale, Kuhn, and Tucker independently of those of the author. An abstract, apparently motivated by consideration of matrices B which have large constant segments, of these results is D. Gale, H. W. Kuhn, and A. W. Tucker, Bull. Amer. Math. Soc. Abstract 55-11-472.