A CHARACTERIZATION OF PLANE LIGHT OPEN MAPPINGS

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1. **Minimal functions.** We are concerned with continuous functions whose domains are open subsets of the plane P and whose ranges are contained in P. In this paper the word *mapping* is used to designate such a function. By a *disk* we mean a subset of P which is a closed topological 2-cell. If S is a simple closed curve in P, we denote by S^* the disk formed by taking the union of S and the interior of S.

Let S be a simple closed curve in P and let f be a mapping whose domain contains S^*. We say that *f is minimal on S^** if $f(S^*) \subset g(S^*)$ for every mapping g whose domain contains S^* and which is such that $f|S = g|S$ (that is, which is such that $f(x) = g(x)$ for each $x \in S$). We define f to be *minimal* if f is minimal on each disk contained in the domain of f.

In this paper we prove that a light mapping is open if and only if it is minimal.

2. **Winding numbers.** We make use of the concept of *winding number* or *topological index.* If f is a mapping whose domain contains a simple closed curve S and $p \in P - f(S)$, then we denote by $W(f, S, p)$ the winding number of f on S with respect to p. Intuitively, as a point x travels once around S in a counter-clockwise direction, $W(f, S, p)$ is the net number of revolutions that the vector from p to $f(x)$ makes about p (a revolution being positive if made in a counter-clockwise direction and being negative if made in a clockwise direction).

The following facts concerning winding numbers are well known and are assumed.

(i) *If $p \in P - f(S)$, then there exists a neighborhood V of p such that $W(f, S, v) = W(f, S, p)$ for each $v \in V$.*

(ii) *If $p \in P - f(S)$, then $f|S$ is homotopic in $P - p$ to a constant if and only if $W(f, S, p) = 0$.* ($f|S$ is the function obtained by restricting the domain of f to S.)

(iii) *If $p \in P - f(S)$, then $f|S$ can be extended to a continuous function on S^* into $P - p$ if and only if $W(f, S, p) = 0$.*

(iv) *If the domain of a mapping f contains a disk S^*, then $f(S^*) \supset f(S) \cup \{p \mid W(f, S, p) \neq 0\}$.*

(v) *If S_1 and S_2 are simple closed curves which have disjoint interiors, $S_1 \cap S_2$ is an arc, and*

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$$S = \mathrm{Cl}\,((S_1 \cup S_2) - (S_1 \cap S_2)),$$

then

$$W(f, S, p) = W(f, S_1, p) + W(f, S_2, p)$$

for each $p \in P - (S_1 \cup S_2)$.

THEOREM 1. *If S is a simple closed curve and f is a mapping whose domain contains S^*, then f is minimal on S^* if and only if*

$$f(S^*) = f(S) \cup \{p \mid W(f, S, p) \neq 0\}. \tag{A}$$

Let g be a mapping on S^* for which $g|S = f|S$. It follows from (iii) that if $p \in P - f(S)$ and $W(f, S, p) \neq 0$, then $p \in g(S^*)$. Thus, if (A) holds, then $f(S^*) \subset g(S^*)$ and f is minimal on S^*.

If (A) does not hold, then it follows from (iv) that there exists $p \in f(S^*) - f(S)$ for which $W(f, S, p) = 0$. By (iii), there exists an extension g of $f|S$ such that $g(S^*) \subset P - p$. It follows that $f(S^*) \not\subset g(S^*)$ and hence that f is not minimal on S^*.

3. Plane light open mappings. A mapping is *open* (or *strongly interior*) if it takes open sets into open sets. A mapping f is *light* if $f^{-1}(y)$ is totally disconnected for each y in the range of f.

THEOREM 2. *A light mapping f is open if and only if it is minimal.*

Suppose that f is light and minimal. Let U be an open subset of the domain of f, and let $y \in f(U)$. Choose $x \in U$ so that $f(x) = y$. Since $f^{-1}(y)$ is totally disconnected, it can be shown that there exists a simple closed curve S in $U - f^{-1}(y)$ such that $x \in S^* \subset U$. Since $y \notin f(S)$, $W(f, S, y)$ is defined. Since $y \in f(S^*) - f(S)$ and f is minimal, it follows from Theorem 1 that $W(f, S, y) \neq 0$. It follows from (i) that there is a neighborhood V of y such that $W(f, S, v) \neq 0$ for each $v \in V$. Now, from (iv), we obtain $V \subset f(U)$. Thus $f(U)$ is an open set and f is an open mapping.

Now suppose that f is light and open. Let S be a simple closed curve for which S^* is contained in the domain of f and let $y \in f(S^*) - f(S)$. It is known (see [1, p. 191])[1] that $f^{-1}(y) \cap S^*$ is a finite set. We let $x_1, \cdots, x_n$ be the points of $f^{-1}(y) \cap S^*$. Locally at each x_i, f is topologically equivalent to a mapping f_i on the disk $\{z \mid z \in P \text{ and } |z| \leq 1\}$, where $f_i(z) = z^{k_i}$ and k_i is a positive integer. (In this connection see [1, p. 198].) It is now easy to see that for each i there is a simple closed curve C_i about x_i for which $C_i^* \subset S^*$ and $|W(f, C_i, y)| = k_i$. Thus the integers $W(f, C_i, y)$ are all nonzero. Interior to C_i there exist

[1] Numbers in brackets refer to the bibliography at the end of the paper.

points at which f is locally a homeomorphism. At such points f is orientation preserving if $W(f, C_i, y)$ is positive and orientation reversing if $W(f, C_i, y)$ is negative. The set L of points of S^* at which f is locally a homeomorphism is a dense connected subset of S^*. The subset M of L consisting of all points at which f is orientation preserving is both open and closed relative to L. Hence $M=L$ or M is the empty set. It follows that the integers $W(f, C_i, y)$ are either all positive or all negative. We may obviously choose the curves C_i so that the disks C_i^* are disjoint and do not intersect S, and we assume that this has been done. We now choose a finite cellular subdivision of S^* in such a way that each C_i^* is a 2-cell of the subdivision. We let $A_1, \cdots, A_m$ be the 2-cells of the subdivision and let B_j be the boundary of A_j. Making use of (v) and induction, we obtain

$$W(f, S, y) = \sum_{j=1}^{m} W(f, B_j, y).$$

Unless $B_j = C_i$ for some i, then $y \notin f(B_j^*)$ and $W(f, B_j, y) = 0$ by (iii). Therefore

$$W(f, S, y) = \sum_{j=1}^{m} W(f, B_j, y) = \sum_{i=1}^{n} W(f, C_i, y).$$

Since the numbers $W(f, C_i, y)$ are all nonzero and are of like sign, we obtain $W(f, S, y) \neq 0$. It now follows from Theorem 1 that f is minimal.

It would be interesting to know whether or not theorems analogous to Theorem 2 exist for higher-dimensional spaces. The author has been unable to prove such theorems.

Bibliography

1. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942.

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