NUMBER OF INTEGERS IN AN ASSIGNED A, $P \leqq x$ AND PRIME TO PRIMES GREATER THAN x^c

V. RAMASWAMI

In the following:

x denotes any real number greater than 1; y, c, t denote real positive numbers; $l = \log(x)$;

d, n, m, v, k denote integers satisfying $n \geqq 0$, $d > 0$, $m > 0$, $0 < v \leqq k$, $(v, k) = 1$;

$A(m, v, k)$ denotes the set of integers $m(v+nk)$ for varying n;

$f(m, v, k, x, c)$ denotes the number of integers in $A(m, v, k)$ less than or equal to x and prime to primes greater than x^c;

$\psi(x, k)$ denotes the number of positive integers less than or equal to x and prime to k;

$F(x, k) = x\psi(k, k)k^{-1} - \psi(x, k)$; $b(m, k)$, $B(v, k)$ denote functions, to be defined in the context, of the integral variables involved;

$\phi(y)$ and $q(y)$ denote the functions defined on page 100 of my paper[1] entitled *The number of integers* $\leqq x$ *and free of prime divisors* $> x^c$, *and a problem of S. S. Pillai*;

$\pi(v, k, x)$ denotes the number of primes less than or equal to x in $A(1, v, k)$;

$P(v, k, x) = \sum_{p \leqq x,\, p \equiv v \pmod k} p^{-1}$, p denoting a prime number;

$\bar{n}(v, k)$ denotes the least positive integer satisfying $n\bar{n}(v,k) \equiv v \bmod k$ for $(n, k) = 1$.

[We note that $|F(x, k)| < \psi(k, k)$.]

Buchstab[2] has proved the result

$$f(1, v, k, x, c) = x\phi(c)k^{-1} + O(xl^{-1/2}).$$

The object of this note is to prove the following considerable improvements on this result:

(i) $f(m, v, k, x, c) = x\phi(c)(mk)^{-1} + b(m, k)xl^{-1}q(c) + O(x^{c}l^{-1}) + O(xl^{-2})$ for $1/2 \leqq c \leqq 1$,

(ii) $f(m, v, k, x, c) = x\phi(c)(mk)^{-1} + b(m, k)xl^{-1}q(c) + O(xl^{-3/2})$ for $c < 1/2$ and $c \geqq 1$, where $b(m, k) = [m\psi(k, k)]^{-1}\int_1^\infty F(t, k)t^{-2}dt$, and the constant implied in each "O" is dependent only on v and k.

It will be noted that for $m = v = k = 1$, this result reduces to Theorem A′ proved in my paper cited above.

The proof follows the lines of that of Theorem A′ with appro-

Received by the editors December 24, 1949.

[1] Duke Math. J. vol. 16 (1949) pp. 99–109.

[2] Doklady Acadamii Nauk SSSR. N.S. vol. 67 (1949) pp. 5–8.

priate modifications, of which the less obvious are indicated below:

(1) we replace Lemma 3 by the known results

(a) $\pi(v, k, x) = [\psi(k, k)]^{-1}$ Li $(x) + O(xl^{-3})$, "O" depending only on k,

(b) $P(v, k, x) = [\psi(k, k)]^{-1} \log\log x + B(v, k) + h(x, v, k)l^{-2}$, where

$$|h(y, v, k)| < (1/24)(1 + 1/2)^2 \text{ for } y \geqq \exp(l^{1/2}) \text{ and } x > a_{10} > 1.$$

(2) We replace Lemma 6 of Theorem A′ by the result (i) above of this theorem. To prove this latter, we observe that on account of the unique factorisation theorem, we have, for $1/2 \leqq c \leqq 1$,

$$f(m, v, k, x, 1) - f(m, v, k, x, c)$$

$$= \sum_{1 \leqq n \leqq x^{1-c}/m;\, (n,k)=1} \left[\pi\left\{\bar{n}(v, k), k, \frac{x}{mn}\right\} - \pi\{\bar{n}(v, k), k, x^c\}\right]$$

$$= \frac{1}{\psi(k, k)} \sum\left[\mathrm{Li}\left(\frac{x}{mn}\right) - \mathrm{Li}(x^c)\right] + O(xl^{-2}),$$

and

$$\sum \mathrm{Li}\left(\frac{x}{mn}\right) = \int_{1-0}^{(x^{1-c})/m+0} \mathrm{Li}\left(\frac{x}{mt}\right) d\psi(t, k)$$

$$= \psi\left[\frac{x^{1-c}}{m}, k\right] \mathrm{Li}\,(x^c) + \frac{x}{m}\int_1^{x^{1-c}/m} \frac{\psi(t, k)}{t^2 \log(x/mt)} dt$$

$$= \psi\left[\frac{x^{1-c}}{m}, k\right] \mathrm{Li}\,(x^c) + \frac{x\psi(k,k)}{mk}\int_1^{x^{1-c}/m} \frac{dt}{t \log(x/t)}$$

$$- \frac{x}{m}\int_1^{x^{1-c}/m} \frac{F(t, k)}{t^2 \log(x/mt)} dt.$$

Hence it follows that

$$f(m, v, k, x, c) = (x/mk)(1 + \log c) + b(m, k)xl^{-1} + O(x^c l^{-1}) + O(xl^{-2}),$$
$$\text{for } 1/2 \leqq c \leqq 1,$$

which is the desired result, since $\phi(c) = 1 + \log c$ for $1/2 \leqq c \leqq 1$, and $q(1) = 0$, and $q(c) = 1$ for $1/2 \leqq c < 1$.

ANDHRA UNIVERSITY