ALGEBRAIC HOMOGENEOUS SPACES OVER FIELDS OF CHARACTERISTIC TWO

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1. **Introduction.** This paper is a sequel to Chow's recent results[1] on algebraic homogeneous spaces. Leaving aside the Grassmann spaces, the spaces considered by Chow are essentially those consisting of the r-dimensional linear varieties in a $(2r+1)$-dimensional projective space S over a field K, which are *self-conjugate* with respect to a basic correlation Δ of the space S, Δ being defined either by an alternate nondegenerate bilinear form (null system) or by a symmetric nondegenerate bilinear form (polar system). Now when K has characteristic 2, both cases merge into a single one. But within the space N_r of self-conjugate varieties with respect to the correlation defined by the bilinear form $\sum_{i=0}^{r} (y_{r+i+1}x_i + y_i x_{r+1+i})$, we may now consider the subspace T_r of the varieties belonging to N_r and in which the quadratic form $\sum_{i=0}^{r} x_{r+i+1}x_i$ vanishes; the basic group of that space is a subgroup of the symplectic group, namely the *orthogonal group* corresponding to the quadratic form just mentioned (or, more precisely, that group enlarged by some semi-linear transformations). We want to show that Chow's characterization of the basic groups of the spaces he considers extends to the basic group of T_r. Since the proofs consist mostly of a mere translation of Chow's proofs to the situation we are considering, we shall suppress most of them, referring the reader to Chow's paper for the missing arguments.[2] There is, however, one point over which we shall go into some detail: it corresponds to Chow's long proof of his Theorem III (see [**3**, pp. 46–49]), and it will be seen that this proof may be appreciably shortened and applied to a more general question, namely the study of the space of the "invariant" (or "totally isotropic") linear varieties of *maximal* dimension corresponding to a polar system when that dimension is *smaller* than r.

2. **Singular subspaces relative to a quadratic form.** In the following, K will denote a field of characteristic 2, E an $(n+1)$-dimensional vector space over K, n being an odd number, $n=2r+1$; g will be a *regular nondefective* quadratic form defined in E, and f the cor-

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[1] See [**3**]; we consider only the "projective" part of Chow's paper. Numbers in brackets refer to the bibliography at the end of the paper.

[2] In order to facilitate the application of Chow's proofs to our results, we have kept his notations as far as possible.

responding alternate bilinear form, which is therefore nondegenerate (see [5, pp. 39–40]). A *singular*[3] subspace V of E is a subspace in which g vanishes identically; such a subspace is totally isotropic (that is, contained in its conjugate with respect to f), but the converse does not hold. The maximum dimension ν of singular subspaces may take any value such that $0 \leqq \nu \leqq r+1$; it is called the *index* of the form g; in the following, *we shall always suppose that* $\nu \geqq 3$, and put $\nu = s+1$ $(s \geqq 2)$.

The following lemmas are known:[4]

LEMMA 1. *Let V and W be two singular subspaces such that every vector of V is conjugate to every vector of W; then $V+W$ is singular.*

LEMMA 2. *Let V be a singular subspace of dimension t. There exists a singular subspace W of dimension t such that $V+W$ is nonisotropic and $V \cap W = \{0\}$. For any such subspace W, there exists a basis $(e_i)(1 \leqq i \leqq 2t)$ of $V+W$, such that $e_1, e_2, \cdots, e_t$ constitute a basis of V, $e_{t+1}, \cdots, e_{2t}$, a basis of W, and one has $f(e_i, e_{t+j}) = \delta_{ij}$ (Kronecker's index).*

LEMMA 3. *Every singular subspace is contained in a singular subspace of maximal dimension.*

From these lemmas we first deduce the following lemma.

LEMMA 4. *The subspace spanned by the union of all singular subspaces of E is E itself.*

Let $a \neq 0$ be a singular vector, H the hyperplane conjugate to a (and containing a), b a vector such that $f(a, b) \neq 0$. The plane P containing a and b is nonisotropic, and therefore (Lemma 2) contains a singular vector e_1 such that $f(a, e_1) = 1$. Let $P' \subset H$ be the $(n-1)$-dimensional nonisotropic subspace conjugate to P, and let (c_i) $(2 \leqq i \leqq n)$ be any basis of P'. One has $f(e_1, a+c_i) = 1$, and therefore, in the plane determined by e_1 and $a+c_i$, which is not isotropic, there exists a second singular vector e_i such that $f(e_1, e_i) = 1$; it is then clear that the $n+1$ singular vectors e_1, e_i (for $2 \leqq i \leqq n$) and a constitute a basis for E.

Let now $S = P(E)$ be the projective n-dimensional space corresponding to E; the linear varieties in S corresponding to the singular subspaces of E will also be called *singular*. The proofs of Chow's Lemmas 1, 2, and 3 may now be repeated without change (owing

[3] These subspaces are called "totally singular" in [5]; we change the terminology slightly for the sake of brevity.

[4] For the proofs, see [5, pp. 40–41].

to Lemmas 1 and 2 above) and give the similar results:

LEMMA 5. *If every element of a set of singular* $[t]$ $(t\leqq s)$ *are* $(t-1)$-*incident, then all the elements of the set must be incident with either one common singular* $[t+1]$ *or one common singular* $[t-1]$; *the first case cannot occur if* $t=s$.

LEMMA 6. *If two singular* $[t]_1$, $[t]_2$ $(t\leqq s)$ *have a* $[u]$ *as their intersection, then there exist two singular* $[s]_1$, $[s]_2$ *containing* $[t]_1$, $[t]_2$ *respectively, and having also* $[u]$ *as their intersection.*

LEMMA 7. *If two singular* $[t]_a$, $[t]_b$ $(t\leqq s)$ *have a* $[t-u]$ *as their intersection, then there exists a sequence of* $u+1$ *singular varieties* $[t]_1=[t]_a$, $[t]_2, \cdots, [t]_{u+1}=[t]_b$, *all incident with the* $[t-u]$, *and such that each two consecutive varieties are* $(t-1)$-*incident.*

3. **Adjacence preserving transformations of** T_s. We shall denote by T_s the set of all singular varieties of maximal dimension s $(\geqq 2)$ in S. The "basic group" of T_s is the group of transformations induced by all collineations of S defined by semi-linear one-to-one transformations u of E such that $g(u(x))=\lambda(g(x))^{\sigma}$, where $\lambda\in K$ and σ is the automorphism of the field K relative to the semi-linear transformation u. With the same definitions of "adjacence" as in Chow's paper, we want to prove the following theorem.

THEOREM 1. *Any one-to-one adjacence preserving transformation of the space* T_s $(s\geqq 2)$ *onto itself is a transformation of the basic group.*

Let Γ be an adjacence preserving one-to-one transformation of T_s. The first part of Chow's proof of his Theorems II and III may be repeated without change, since it relies only on his Lemmas 1, 2, and 3; this argument extends the given transformation "downward" to *all* singular varieties of S (of dimension ranging from 0 to its maximum s), the extended transformation Γ being one-to-one and preserving incidence relations. Still following Chow, we notice that if Γ transforms a singular $[s]$ into $[s]'$, it induces a *collineation* of $[s]$ onto $[s]'$, and that moreover all these collineations involve the same automorphism of the ground field K. There remains only the last part of the proof which consists in showing that all these collineations are induced by a single collineation Θ of the entire space S.

To do this, we build an ascending chain of spaces $[s+k]_k$, starting from a singular $[s]_0$, and ending with S (for $k=n-s$), and we define inductively Θ in each $[s+k]_k$ in succession. Our procedure requires a special treatment of the two first steps of the induction.

1°. From Lemma 4, there is a singular point $[0]_1$ which does not belong to $[s]_0$; we define $[s+1]_1$ as the join of $[s]_0$ and $[0]_1$. The hyperplane conjugate to $[0]_1$ cannot contain $[s]_0$, for there are no $(s+1)$-dimensional singular varieties (Lemma 1); it therefore intersects $[s]_0$ along a singular $[s-1]_a$. The join $[s]_1=[s-1]_a\cup[0]_1$ is singular; since the restrictions of Γ to $[s]_0$ and $[s]_1$ coincide on $[s-1]_a$, we may extend Γ to a unique collineation Θ of $[s+1]_1$ (onto another $(s+1)$-dimensional linear variety of S) coinciding with Γ on $[s]_0$ and $[s]_1$. Now there are no singular points in $[s+1]_1$ other than those on $[s]_0$ and $[s]_1$, for if $[0]$ were such a point, the line joining it to a point $[0]'$ in $[s]_1$ but not in $[s-1]_a$ would meet $[s]_0$ in a third singular point $[0]''$, hence would be a singular line conjugate to $[s-1]_a$; but this would mean (Lemma 1) that $[s+1]_1$ is singular, which is absurd. Therefore Θ and Γ coincide for every singular point in $[s+1]_1$.

2°. Let $[0]_2$ be a singular point which is not in the $(n-s)$-dimensional variety conjugate to $[s-1]_a$ (Lemma 4). The hyperplane conjugate to $[0]_2$ intersects $[s]_0$ along a singular $[s-1]_b$ *distinct* from $[s-1]_a$, and $[s]_2=[s-1]_b\cup[0]_2$ is singular. Since Θ and Γ (defined respectively on $[s+1]_1$ and $[s]_2$) coincide on $[s-1]_b$, we may extend Θ to a unique collineation Θ of $[s+2]_2=[s+1]_1\cup[0]_2$ which coincides with Γ on $[s]_2$. We have to prove that it coincides also with Γ on *every* singular point contained in $[s+2]_2$. Let $[0]$ be such a point (not in $[s+1]_1$ nor in $[s]_2$), and suppose first that the hyperplane conjugate to $[0]$ intersects $[s]_0$ along an $[s-1]$ *distinct* from $[s-1]_b$; it intersects therefore $[s]_2$ along an $[s-1]'$ which is also distinct from $[s-1]_b$; hence there is in $[s-1]'$ a singular line $[1]'$ which is not in $[s+1]_1$. The plane $[2]=[0]\cup[1]'$ is then singular and intersects $[s+1]_1$ along a singular line $[1]$ distinct from $[1]'$. Since Θ coincides with Γ on $[1]$ and $[1]'$, it also coincides with Γ on $[2]$, hence on $[0]$.

To obtain a point $[0]$ having the preceding property, consider the intersection $[s-1]_1$ of the hyperplane conjugate to $[0]_2$ and of $[s]_1$, and any point $[0]_c$ of $[s]_3=[0]_2\cup[s-1]_1$ not on $[s]_2$; there are such points, for if $[s]_2$ and $[s]_3$ coincided, the line joining any point of $[s-1]_1$ (not on $[s-1]_a$) and any point of $[s-1]_b$ (not on $[s-1]_a$) would be singular, contrary to what we have seen in 1°. Then $[s-1]_b$ cannot be conjugate to $[0]_c$, for otherwise the line joining $[0]_c$ and $[0]_2$ would be singular and conjugate to $[s-1]_b$, and its join with $[s-1]_b$ would be a singular $(s+1)$-dimensional variety, which is absurd. It is clear moreover that every point of the singular variety joining $[0]_c$ to the intersection $[s-1]_c$ of its conjugate hyperplane with $[s]_0$ has the same property as $[0]_c$, and therefore Θ and Γ coincide on that singular variety $[s]_4$.

If now $[0]$ is a singular point in $[s+2]_2$ conjugate to $[s-1]_b$, it cannot be also conjugate to $[s-1]_c$ (since it is not in $[s]_0$); we may therefore argue as before, replacing $[s]_2$ by $[s]_4$, and we have thus verified that Θ and Γ coincide on every singular point in $[s+2]_2$.

3°. We now define, for $k\geqq 3$, $[s+k]_k$ as the join of $[s+k-1]_{k-1}$ and a singular point $[0]_k$ not in $[s+k-1]_{k-1}$ (Lemma 4). We consider the intersection $[s-1]$ of $[s]_0$ with the hyperplane conjugate to $[0]_k$, and then extend Θ to $[s+k]_k$ in exactly the same fashion as in 2°. To prove next that Θ thus extended coincides with Γ on every singular point contained in $[s+k]_k$, we may also proceed exactly as in 2° (replacing of course $[0]_2$ by $[0]_k$), if $[s-1]$ is distinct from $[s-1]_a$. If, on the contrary, $[s-1]$ is identical with $[s-1]_a$, it is distinct from $[s-1]_b$, and we may then replace $[s]_1$ by $[s]_2$ in the second part of the argument in 2° ($[0]_2$ always being replaced by $[0]_k$).

The collineation Θ may thus be extended to the whole space S, and since it coincides with Γ on the set of all singular points, it transforms a singular point into a singular point. We want to prove that Θ commutes with the correlation Δ defined by f, which means that it transforms two conjugate points into conjugate points. Now this is clear if both points are singular, for the line joining them is then singular, hence transformed by Θ into a singular line. Let us then consider a nonsingular point $[0]$ and the conjugate hyperplane $[n-1]$ to $[0]$. Let $[n-2]$ be a nonisotropic linear variety in $[n-1]$ which does not contain $[0]$ and is conjugate to a nonisotropic line joining $[0]$ to a singular point $[0]_1$ not contained in $[n-1]$. Since $s\geqq 2$, $[n-2]$ contains singular points (conjugate to $[0]_1$) by Lemma 3, hence is spanned by the singular points it contains (Lemma 4). Let $[0]'$ be any of these points; since it is conjugate to $[0]$, the line joining it to $[0]$ is isotropic but not singular, hence contains *only one* singular point $[0]'$ (see [**5**, p. 40]); it is therefore transformed by Θ into a line having the same property, hence is isotropic, which means that $[0]'$ is transformed by Θ into a point conjugate to the point image of $[0]$ by Θ. This, together with the preceding remarks, shows that Θ transforms conjugate points into conjugate points. From this property, and the fact that Θ transforms singular points into singular points, a simple argument (see [**6**, §46]) yields finally Theorem 1.

4. **Transvections and rotations.** It is a little known result[5] that, over a field K of characteristic 2, the orthogonal group $\mathfrak{O}_{n+1}(K, g)$

[5] For finite fields K, this result is proved in [**4**, p. 206], and in [**7**, p. 339].

$(n=2r+1)$ possesses an invariant subgroup $\mathfrak{O}^+_{n+1}(K, g)$ which is the exact counterpart of the group of rotations (that is, orthogonal transformations of determinant $+1$) in the orthogonal group over a field of characteristic not 2. We shall give in this section a simple proof of that theorem.

Suppose first that g is a form of maximal index $\nu=r+1$, and let V and W be two singular subspaces of E of maximal dimension. We recall (see [**5**, pp. 41–42]) that a *transvection* in $\mathfrak{O}_{n+1}(K, g)$ is an (involutive) orthogonal transformation u such that $u(x) =x+af(x, a)/g(a)$, where a is any *nonsingular* vector of E. We are going to prove that, if k is the dimension of $V\cap W$, *the dimension of* $V\cap u(W)$ *is* $k+1$ *or* $k-1$. Suppose first that a is conjugate to $V\cap W$, which means that $a\in V+W$. Any vector y belonging to $V\cap u(W)$ is then in the intersection of V with the $(\nu+1)$-dimensional subspace U generated by W and a. Conversely, if $y\in U\cap V$ and $y\notin W$, the plane P generated by a and y is contained in U, hence intersects W along a line D; P contains two distinct singular lines D and $D' =Ky$, and a nonsingular line Ka; it is therefore a nonisotropic plane, invariant by the transvection u. But it is clear that u may not leave invariant any singular line not conjugate to a; therefore it exchanges the lines D and D', which proves that $U\cap V= V\cap u(W)$, and $U\cap V$ is obviously $(k+1)$-dimensional. Next consider the case in which a is not conjugate to $V\cap W$, and therefore $a\notin V+W$; then no vector $x\in W$ which is not conjugate to a may be sent by u into a vector of V, since the plane defined by $u(x)$ and x would contain a and be contained in $V+W$; the intersection $V\cap u(W)$ is then identical with the intersection of $V\cap W$ with the hyperplane conjugate to a, and therefore is $(k-1)$-dimensional.

From this result we deduce at once that *no product of an odd number of transvections may be the identity in* $\mathfrak{O}_{n+1}(K, g)$, for if v is such a product, the dimension of $V\cap v(W)$ differs from that of $V\cap W$ by an *odd* number. As every orthogonal transformation is a product of transvections, we thus see that those orthogonal transformations which are *products of an even number of transvections* constitute a *normal subgroup* $\mathfrak{O}^+_{n+1}(K, g)$ of index 2 in $\mathfrak{O}_{n+1}(K, g)$, which we may again call the *group of rotations*.

This result may easily be extended to *any quadratic form* g over E. We have only to consider a suitable algebraic extension K_1 of K, such that g has maximal index $r+1$ *over* K_1; the group $\mathfrak{O}_{n+1}(K, g)$ is then a subgroup of $\mathfrak{O}_{n+1}(K_1, g)$, and transvections in the first are also transvections in the second; hence no product of an odd number of transvections in $\mathfrak{O}_{n+1}(K, g)$ may be the identity in that group.[6]

Next let us study the way in which the group $\mathfrak{O}^+_{n+1}(K, g)$ transforms singular subspaces. If V and W are two singular subspaces of the same dimension k, it follows from Arf's theorem (see [1]) that there is always an orthogonal transformation sending V into W. If $k \leqq r$, there is also a rotation sending V into W, for there are transvections leaving invariant V (for instance, transvections corresponding to nonsingular vectors conjugate to V). But if $k = r+1$ (which means $\nu = r+1$), the set T_r of k-dimensional singular subspaces splits into two subsets U_r, U'_r which are homogeneous spaces for the group of rotations $\mathfrak{O}^+_{n+1}(K, g)$; if V_0 is any $(r+1)$-dimensional singular subspace, we may define U_r (resp. U'_r) as consisting of the subspaces W of T_r such that the difference between the dimensions of V_0 and $V_0 \cap W$ is an *even* (resp. *odd*) number. It is then clear that any orthogonal transformation which sends a subspace of U_r (resp. U'_r) into another subspace of U_r (resp. U'_r) is a rotation, and that no rotation may change a subspace of U_r into a subspace of U'_r. This, together with Arf's theorem, proves that if V and W are any two subspaces belonging both to U_r (resp. U'_r) the difference between the dimensions of V and $V \cap W$ is an even number; on the contrary, when V belongs to U_r and W to U'_r, the difference between the dimensions of V and $V \cap W$ is an odd number (this is proved simply by sending V into V_0 by a rotation).

All these results parallel closely the well known facts concerning the space N_r of self-conjugate varieties of maximum dimension in a polar system over a field of characteristic not 2. They enable one to state and prove the following theorem, corresponding to Chow's Theorem VII.

THEOREM 2. *Any one-to-one adjacence-preserving transformation of the space U_r $(r \geqq 4)$ onto itself is a transformation of the basic group of that space.*

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[6] For a given orthogonal transformation u, the fact that it belongs to the group of rotations may be ascertained by computing its *Dickson invariant* [**4**, p. 206]. Suppose, for simplicity's sake, that $\nu = r+1$, and (e_i) is a basis of E satisfying the conditions of Lemma 2 (for $t = r+1$). Put $u(e_i) = \sum_{j=1}^{r+1} \alpha_{ij} e_j + \sum_{j=1}^{r+1} \beta_{ij} e_{r+j+1}$ and $u(e_{r+i+1}) = \sum_{j=1}^{r+1} \gamma_{ij} e_j + \sum_{j=1}^{r+1} \delta_{ij} e_{r+j+1}$; the Dickson invariant is then $\Delta(u) = \sum_{i=1}^{r+1} \sum_{j=1}^{r+1} \beta_{ij}\gamma_{ij}$; multiplying u with a transvection changes $\Delta(u)$ into $\Delta(u)+1$. Therefore, one has $\Delta(u) = 0$ for rotations, and $\Delta(u) = 1$ for the other orthogonal transformations (products of an odd number of transvections). For quadratic forms g of any index, the Dickson invariant may be defined similarly, by going over to an extension K_1 of K, in which g has index $r+1$.

Here of course, "adjacence" is to be understood in Chow's sense and the basic group of U_r is simply the subgroup (of index 2) of the basic group of T_r preserving U_r.

We suppress the proof, which is simply a restatement of Chow's proof of his Theorem VII.

5. **Spinors and triality.** The case $r=3$ is exceptional in Chow's Theorem VII, due to the well known Study-Cartan "triality" (see [2] and [8]) between U_r, U_r' and the space S itself when $r=3$. Now this also extends perfectly to the case in which K has characteristic 2.

One of the best ways to see this is probably to carry over to that case the classical theory of *spinors*. We shall give a brief outline of that extension. We suppose g has index $\nu=r+1$; then there exists a basis (e_i) $(1 \leqq i \leqq 2r+2)$ of E, such that the e_i of index less than or equal to $r+1$ (resp. greater than $r+1$) span a singular subspace V (resp. W) of dimension $r+1$, and one has $f(e_i, e_{r+j+1})=\delta_{ij}$ for $1 \leqq i \leqq r+1$, $1 \leqq j \leqq r+1$. The *Clifford algebra* $C(g)$ of the form g over K is then defined (see [1]) as the associative algebra generated by the unit element $c_0=1$ of K, and $2r+2$ linearly independent elements c_i $(1 \leqq i \leqq 2r+2)$ such that

$$
(1)\quad
\begin{aligned}
&c_i^2 = 0 && (1 \leqq i \leqq 2r+2),\\
&c_j c_i = c_i c_j = 0 && (1 \leqq i \leqq r+1,\ 1 \leqq j \leqq r+1),\\
&c_{r+1+i} c_{r+1+j} = c_{r+1+j} c_{r+1+i} = 0 && (1 \leqq i \leqq r+1,\ 1 \leqq j \leqq r+1),\\
&c_i c_{r+j+1} + c_{r+j+1} c_i = \delta_{ij} && (1 \leqq i \leqq r+1,\ 1 \leqq j \leqq r+1).
\end{aligned}
$$

It is immediately verified that for $r=0$, $C(g)$ is isomorphic to the total algebra of matrices of order 2 over K(c_1, c_2, c_1c_2, and c_2c_1 satisfying the same relations as the canonical basis of that algebra); and for an arbitrary r, $C(g)$ is isomorphic with the Kronecker product of $r+1$ such algebras of matrices, hence is isomorphic to the total algebra of matrices of order 2^{r+1} over K. Now, if u is any transformation of the orthogonal group $\mathfrak{O}_{n+1}(K, g)$, the fact that u leaves g invariant implies that it may be extended in the usual way to an ***automorphism*** of the Clifford algebra $C(g)$; if $u(e_i)=\sum_{j=1}^{n+1} \alpha_{ij} e_j$, one has merely to define $u(c_i)=\sum_{j=1}^{n+1} \alpha_{ij} c_j$, and it is at once verified that the elements $u(c_i)$ verify the same relations (1) as the c_i themselves. Since this automorphism u of $C(g)$ leaves invariant every element of the center K of that algebra, the Skolem-Noether theorem shows that it is an *inner* automorphism $z \rightarrow s(u)z(s(u))^{-1}$, where the element $s(u) \in C(g)$ is determined up to a scalar factor.[7]

From now on, we *identify* c_i and e_i $(1 \leqq i \leqq n+1)$ so that E appears as a subspace of the algebra $C(g)$. It is then readily verified that for any two vectors x, y in E, one has (in $(C(g))$ $xy+yx=f(x, y)$, and $x^2=g(x)$. From that result it follows that if a is any nonsingular vector in E, and u the transvection $u(x)=x+af(x, a)/g(a)$, then the extension of u to $C(g)$ is the inner automorphism $z \rightarrow aza^{-1}$, for this is immediate when $z \in E$, and E generates the algebra $C(g)$.

Following Cartan (see [2, p. 9]), we now define the *spinors* by concretizing the elements of $C(g)$ as particular matrices of order 2^{r+1}. The spinor space F is a 2^{r+1}-dimensional space over K, in which we consider a basis (a_L) which we index with the 2^{r+1} subsets L of the set of the integers $1, 2, \cdots, r+1$. For $1 \leqq i \leqq r+1$, let H_i be the matrix such that

$$(2) \qquad \begin{aligned} H_i \cdot a_L &= a_{L \cup \{i\}} && \text{if } i \notin L, \\ H_i \cdot a_L &= 0 && \text{if } i \in L; \end{aligned}$$

similarly, let H_{r+1+i} be the matrix such that

$$(3) \qquad \begin{aligned} H_{r+1+i} \cdot a_L &= a_{L-\{i\}} && \text{if } i \in L, \\ H_{r+1+i} \cdot a_L &= 0 && \text{if } i \notin L. \end{aligned}$$

Then it is readily verified that the matrices H_i $(1 \leqq i \leqq 2r+2)$ satisfy the same relations (1) as the basis (c_i) of the subspace E of $C(g)$; since the algebra $C(g)$ is simple, it follows that these matrices generate the whole algebra of matrices of order 2^{r+1} over K, and that the correspondence $c_i \rightarrow H_i$ defines a faithful representation of $C(g)$ onto that algebra. To a vector $x=\sum_{i=1}^{n+1} \alpha_i e_i$ in E corresponds then the matrix $X=\sum_{i=1}^{n+1} \alpha_i H_i$, and one has $X^2=g(x)I$. Let $t \neq 0$ be any spinor in F; the vectors $x \in E$ such that $X \cdot t=0$ constitute a singular subspace in E; the spinor t is said to be *simple* if that subspace has maximal dimension $r+1$. For instance the spinor $a_{12 \cdots (r+1)}$ is simple, the corresponding singular subspace being generated by the e_i of index less than or equal to $r+1$; and it is readily proved that *any* simple spinor may be deduced from that particular one by application of an orthog-

[7] When K is a *perfect* field (which here means that every element in K is a square) the element $s(u)$ may be *normalized* in the usual way, so that $u \rightarrow s(u)$ is a *one-valued* representation of the group $\mathfrak{O}_{n+1}(K, g)$ into $C(g)$. This is in sharp contrast with the case of spinors over a field of characteristic not 2, where the normalization of $s(u)$ yields only the well known *double-valued* spinor representation of the orthogonal group.

onal transformation (identified, up to a scalar factor, with a matrix of $C(g)$) (see [2, p. 21]).

Consider now in F the subspaces F^+ and F^- generated by the a_L having an *even* (resp. *odd*) number of indices. The relations (2) and (3) make it clear that any transvection transforms F^+ into F^- and conversely; hence any *rotation* (§4) leaves invariant both F^+ and F^-, the elements of which are called respectively *even* and *odd semi-spinors*. The *simple* semi-spinors correspond respectively to both classes U_r, U_r' of singular subspaces defined in §4.

Now, when $r=3$, the three spaces E, F^+, and F^- have the same dimension 8, and it is easy to define, after Cartan (see [2, p. 53]), a one-to-one linear transformation of E onto F^+ (or F^-) which sends singular vector into simple semi-spinors. Using the relation between simple semi-spinors and singular subspaces of U_r (or U_r'), it is a matter of simple computation to verify that this transformation yields a transformation between S and U_r, sending singular points on a same singular line into singular subspaces having a singular $[r-2]$ in common; the verification needs be done only for a particular singular line, for instance that defined by e_1 and e_2; we suppress the details. This is of course the Study-Cartan "triality," which is thus extended to fields of characteristic 2.

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