A THEOREM IN THE GEOMETRY OF MATRICES

JOSEPHINE MITCHELL

The following theorem is of interest in that it shows a connection between a result due to Hua in the geometry of matrices with one due to Bergman in several complex variables. Let $Z=(z_{jk})$ $(j=1, \cdots, m;$ $k=1, \cdots, n)$ be a matrix of signature (m, n) (that is, of m rows and n columns), $\bar{Z}'=(\bar{z}_{kj})$ its conjugate transposed matrix, I the identity matrix, and 0 the zero matrix. We consider the domain defined by $I-Z\bar{Z}'>0$, consisting of all points Z with mn complex coordinates $(z_{11}, z_{12}, \cdots, z_{mn})$ for which the Hermitian quadratic form,

$$\sum_{j,k=1}^{m}\left(\delta_{jk} - \sum_{i=1}^{n} z_{ji}\bar{z}_{ki}\right) x_j\bar{x}_k,$$

is positive definite in the auxiliary variables $(x_1, \cdots, x_n)$. The domain $I-Z\bar{Z}'>0$ lies in $2mn$-dimensional Euclidean space and is a certain generalization of the unit circle in the complex z-plane.

We prove that:

THEOREM. *The positive definite quadratic differential form*

$$Q \equiv \sum_{j,p=1}^{m}\sum_{k,q=1}^{n} \frac{\partial^2 \log\left[\det (I - Z\bar{Z}')\right]^{-1}}{\partial z_{jk}\partial \bar{z}_{pq}} dz_{jk}d\bar{z}_{pq}$$

defines a metric in the domain $I-Z\bar{Z}'>0$ *which is invariant with respect to the conjunctive group of signature* (m, n).

The conjunctive group of signature (m, n) is the set of matrices T such that $\bar{T}FT'=F$, where

$$F = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad\text{and}\quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

PROOF OF THE THEOREM. Hua has proved that *the positive definite quadratic differential form* $\sigma((I-Z\bar{Z}')^{-1}dZ(I-\bar{Z}'Z)^{-1}d\bar{Z}')$, where $\sigma(Z)$ is the trace of matrix Z and dZ its differential, *is invariant with respect to the conjunctive group of signature* (m, n) [*On the theory of automorphic functions of a matrix variable* I, Amer. J. Math. vol. 66 (1944) pp. 470–488]. We prove the theorem by proving the following formula, using techniques due to Hua

$$\text{(A)}\qquad Q = \sigma((I - Z\bar{Z}')^{-1}dZ(I - \bar{Z}'Z)^{-1}d\bar{Z}').$$

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Now $d(\det X)=\sum_{k=1}^{n} dx_{1k}X_{1k}+\cdots+\sum_{k=1}^{n} dx_{nk}X_{nk}=\sigma(dXX^*)$, where X_{jk} is the cofactor of x_{jk} in the matrix X and X^* is the adjoint matrix. Also, for nonsingular matrices $XX^*=I \det X$ or $X^*=(\det X)X^{-1}$. Hence, since $\sigma(kA)=k\sigma(A)$ for k scalar,

$$d(\det X) = \det X\sigma(dXX^{-1}), \tag{1}$$

and

$$d(\log \det X) = \sigma(dXX^{-1}). \tag{2}$$

We now form Q in two steps. Using (2), we have

$$\begin{aligned} -\sum_{p=1}^{m}\sum_{q=1}^{n} \frac{\partial}{\partial \bar{z}_{pq}} \log\det(I-Z\bar{Z}')d\bar{z}_{pq} &= -d\bar{Z}(\log\det(I-Z\bar{Z}')) \\ &= -\sigma(d\bar{Z}(I-Z\bar{Z}')(I-Z\bar{Z}')^{-1}) \\ &= \sigma(Zd\bar{Z}'(I-Z\bar{Z}')^{-1}); \end{aligned} \tag{3}$$

whence

$$\begin{aligned} Q &= -\sum_{j=1}^{m}\sum_{k=1}^{n} \frac{\partial}{\partial z_{jk}}\left[\sum_{p=1}^{m}\sum_{q=1}^{n} \frac{\partial}{\partial \bar{z}_{pq}} \log\det(I-Z\bar{Z}')d\bar{z}_{pq}\right]dz_{jk} \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{\partial}{\partial z_{jk}} \sigma(Zd\bar{Z}'(I-Z\bar{Z}')^{-1})dz_{jk}. \end{aligned} \tag{4}$$

Writing $X=Zd\bar{Z}'$, $Y=(I-Z\bar{Z}')^{-1}$ and using the definition of $\sigma(A)$, we have

$$\begin{aligned} Q &= \sum_{j,k} \frac{\partial}{\partial z_{jk}} \sigma(XY)dz_{jk} \\ &= \sum_{j,k} \frac{\partial}{\partial z_{jk}}\left(\sum_{\alpha,\beta} x_{\alpha\beta}y_{\beta\alpha}\right)dz_{jk} (X=(x_{\alpha\beta}),\ Y=(y_{\alpha\beta})) \\ &= \sum_{j,k}\sum_{\alpha,\beta}\left(\frac{\partial x_{\alpha\beta}}{\partial z_{jk}} y_{\beta\alpha} + x_{\alpha\beta}\frac{\partial y_{\beta\alpha}}{\partial z_{jk}}\right)dz_{jk} \\ &= \sum_{\alpha,\beta}\left[\left(\sum_{j,k}\frac{\partial x_{\alpha\beta}}{\partial z_{jk}}dz_{jk}\right)y_{\beta\alpha} + x_{\alpha\beta}\left(\sum_{j,k}\frac{\partial y_{\beta\alpha}}{\partial z_{jk}}dz_{jk}\right)\right] \\ &= \sum_{\alpha,\beta}(dx_{\alpha\beta}y_{\beta\alpha} + x_{\alpha\beta}dy_{\beta\alpha}) \\ &= \sigma(dXY + XdY). \end{aligned} \tag{5}$$

But $dX=dZd\bar{Z}'$ and $dY=-(I-Z\bar{Z}')^{-1}(-dZ)\bar{Z}'(I-Z\bar{Z}')^{-1}$ so that

$$
(6)\quad \begin{aligned} Q &= \sigma(dZd\bar{Z}'(I - Z\bar{Z}')^{-1} + Zd\bar{Z}'(I - Z\bar{Z}')^{-1}dZ\bar{Z}'(I - Z\bar{Z}')^{-1}) \\ &= \sigma(dZd\bar{Z}'(I - Z\bar{Z}')^{-1} + dZ\bar{Z}'(I - Z\bar{Z}')^{-1}Zd\bar{Z}'(I - Z\bar{Z}')^{-1}), \end{aligned}
$$

since $\sigma(A+B)=\sigma(A)+\sigma(B)$ and $\sigma(AB)=\sigma(BA)$, and this latter term equals

$$
(dZ[I + \bar{Z}'(I - Z\bar{Z}')^{-1}Z]dZ'(I - Z\bar{Z}')^{-1}).
$$

From the identity $I+\bar{Z}'(I-Z\bar{Z}')^{-1}Z=(I-\bar{Z}'Z)^{-1}$ this gives

$$
(7)\quad \begin{aligned} Q &= \sigma(dZ(I - \bar{Z}'Z)^{-1}d\bar{Z}'(I - Z\bar{Z}')^{-1}) \\ &= \sigma((I - Z\bar{Z}')^{-1}dZ(I - \bar{Z}'Z)^{-1}d\bar{Z}'), \end{aligned}
$$

which proves formula(A).

Bergman proved the following theorem. *Let* $K(z_1, \cdots, z_\kappa; \bar{z}_1, \cdots, \bar{z}_\kappa)$ *be the kernel function of a finite schlicht domain lying in* 2κ-*dimensional Euclidean space. Then the Hermitian differential form*

$$
\sum_{\mu,\nu=1}^{\kappa} \frac{\partial^2 \log K(z_1, \cdots, z_\kappa; \bar{z}_1, \cdots, \bar{z}_\kappa)dz_\mu d\bar{z}_\nu}{\partial z_\mu \partial \bar{z}_\nu}
$$

defines a positive definite metric which is invariant with respect to pseudo-conformal transformations. [*Sur les fonctions orthogonales de plusieurs variables complexes*, Mémorial des Sciences Mathématiques, vol. 106, 1947].

A comparison of the two theorems leads us to suspect that the function $V^{-1}[\det (I-Z\bar{Z}')]^{-m-n}$, where V is the Euclidean volume of the domain $I-Z\bar{Z}'>0$, is the Bergman kernel function of the domain $I-Z\bar{Z}'>0$. We have verified this supposition in the case of matrices of signature (2, 2) by the classical method of constructing a complete orthonormal system $\{\phi_p(Z)\}$ $(p=1, 2, \cdots)$ of functions, analytic and Lebesgue square integrable on $I-Z\bar{Z}'>0$, and showing that the absolutely and uniformly convergent series $\sum_{p=1}^{\infty}\phi_p(Z)\bar{\phi}_p(Z)$ has the sum $(12\pi^4)^{-1}[\det (I-Z\bar{Z}')]^{-4}$ [To appear in the Proceedings of the Third Canadian Mathematical Congress, Vancouver, 1949].

UNIVERSITY OF ILLINOIS