NOTES ON FOURIER ANALYSIS. XXX ON THE ABSOLUTE CONVERGENCE OF CERTAIN SERIES OF FUNCTIONS¹

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1. **Introduction.** Recently O. Szász [4]² has proved the following theorems which are generalizations of Fatou's theorem (cf. Zygmund [5, p. 132]).

THEOREM A. *If the series*

$$\sum_{n=1}^{\infty} a_n \cos nx, |a_{n+1}| \leq c|a_n| \quad (c > 0); n = 1, 2, \dots,$$

is absolutely convergent at a point x_0 , then $\sum_{n=1}^{\infty} |a_n| < \infty$. The same is true for the series $\sum_{n=1}^{\infty} a_n \sin nx$, provided that $x_0 \not\equiv 0 \pmod{\pi}$.

THEOREM B. *Let $\phi(x)$ be Riemann integrable in the interval $(0, 1)$ and periodic with period 1 and $\int_0^1 |\phi(x)| dx \neq 0$. If the series*

$$\sum_{n=1}^{\infty} a_n \phi(nx), 0 < |a_{n+1}| < (1 + c/n)|a_n| \quad (c > 0); n = 1, 2, \dots,$$

is absolutely convergent at an irrational point x_0 , then $\sum_{n=1}^{\infty} |a_n| < \infty$.

There are some gaps between the conditions for the sequences of coefficients $\{a_n\}$ in these two theorems. In §2, we shall prove that the conditions for $\{a_n\}$ in Theorem B can be replaced by those in Theorem A.

On the other hand Reves and Szász [2] have proved the analogues of Cantor-Lebesgue's theorem and Denjoy-Lusin's theorem for the double trigonometric series. In §3, we shall generalize these theorems: the former is on the line of Mazur-Orlicz' generalization for Cantor-Lebesgue's theorem [1, Theorem 1 and its corollary] and the latter is in the direction of Salem's generalization for Denjoy-Lusin's theorem [3, Theorem X].

2. **Theorems of Fatou-Szász type.** We shall give here the generalizations of Theorems A and B.

THEOREM 1. *Let $\phi(x)$ be defined in the interval $(0, 1)$ and periodic with period 1 and suppose that there exists an interval $I = (a, b)$,*

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² Numbers in brackets refer to the literature at the end of the paper.

$0 \leq a < b \leq 1$, on which $\phi(x)$ is greater than a positive constant d . If the series $\sum_{n=1}^{\infty} a_n \phi(nx)$ ($a_n \geq 0$) is absolutely convergent at an irrational point ξ and if $0 < a_{n+1} \leq ca_n$ ($c > 0$; $n = 1, 2, \dots$), then $\sum_{n=1}^{\infty} a_n < \infty$.

For the proof we need the following lemma.

LEMMA 1. Let J be an interval and θ be an irrational number contained in the interval $(0, 1)$. Then for an arbitrary real number α there exists a positive integer N independent of α such that at least one of the numbers $\alpha + n\theta$, $n = 1, 2, \dots, N$, is contained in the interval J with modulus 1.

PROOF. We shall prove more precisely that if we denote the number of the points in the sequence $\{\alpha + \nu\theta\}$, $\nu = 1, 2, \dots, n$, which are contained with modulus 1 in the interval J by $n_J(\alpha)$, then the ratio $n_J(\alpha)/n$ tends independently on α to the length of the interval J as $n \rightarrow \infty$; that is, for any given $\epsilon > 0$, there is a positive integer N independent of α such that $|n_J(\alpha)/n - l| < \epsilon$ for $n > N$, where l is the length of the interval J .

To prove this, let $f(x)$ be the function which is obtained by extending the characteristic function of the interval J with period 1. Then there exist two trigonometrical polynomials $p(x)$ and $P(x)$ which satisfy the inequalities

$$(1) \quad p(x) \leq f(x) \leq P(x) \quad \text{in } (0, 1),$$

$$(2) \quad \int_0^1 [f(x) - p(x)] dx < \epsilon/2, \quad \int_0^1 [P(x) - f(x)] dx < \epsilon/2.$$

We can write

$$(3) \quad p(x) = \sum_{k=-m}^m c_k e^{2\pi i k x}, \quad P(x) = \sum_{k=-m}^m C_k e^{2\pi i k x}.$$

Remembering that

$$(4) \quad \frac{n_J(\alpha)}{n} = \frac{1}{n} \sum_{\nu=1}^n f(\alpha + \nu\theta),$$

we have

$$(5) \quad \begin{aligned} \frac{1}{n} \sum_{\nu=1}^n p(\alpha + \nu\theta) &\leq \frac{n_J(\alpha)}{n} \\ &\leq \frac{1}{n} \sum_{\nu=1}^n P(\alpha + \nu\theta) \end{aligned}$$

by (1).

On the other hand

$$\begin{aligned}
 \frac{1}{n} \sum_{\nu=1}^n p(\alpha + \nu\theta) &= \frac{1}{n} \sum_{\nu=1}^n \sum_{k=-m}^m c_k e^{2\pi i k(\alpha + \nu\theta)} \\
 &= \frac{1}{n} \sum_{\nu=1}^n \sum_{k=-m}^m c_k e^{2\pi i k\alpha} e^{2\pi i k\nu\theta} \\
 (6) \quad &= c_0 + \frac{1}{n} \sum'_{k=-m}^m c_k e^{2\pi i k\alpha} \sum_{\nu=1}^n e^{2\pi i k\nu\theta} \\
 &= c_0 + \frac{1}{n} \sum'_{k=-m}^m c_k e^{2\pi i k\alpha} \frac{e^{2\pi i k\theta} - e^{2\pi i k(n+1)\theta}}{1 - e^{2\pi i k\theta}},
 \end{aligned}$$

where $\sum'$ denotes the summation omitting the term for $k=0$. Thus we have

$$\begin{aligned}
 \left| \int_0^1 p(x) dx - \frac{1}{n} \sum_{\nu=1}^n p(\alpha + \nu\theta) \right| &\leq \frac{1}{n} \left| \sum'_{k=-m}^m c_k e^{2\pi i k\alpha} \frac{e^{2\pi i k\theta} - e^{2\pi i k(n+1)\theta}}{1 - e^{2\pi i k\theta}} \right| \\
 (7) \quad &\leq \frac{1}{n} \sum'_{k=-m}^m |c_k| \frac{2}{|1 - e^{2\pi i k\theta}|} \\
 &\leq \frac{2\Delta}{n} \sum'_{k=-m}^m |c_k|,
 \end{aligned}$$

where

$$(8) \quad \Delta = \max_{-m \leq k \leq m, k \neq 0} \frac{1}{|1 - e^{2\pi i k\theta}|}$$

and Δ is finite since θ is irrational and m is fixed; moreover it is evident from the definition that Δ is independent of α .

Therefore if we choose N_1 such that

$$(9) \quad N_1 > 4\Delta \sum'_{k=-m}^m |c_k| / \epsilon,$$

we have

$$(10) \quad \left| \int_0^1 p(x) dx - \frac{1}{n} \sum_{\nu=1}^n p(\alpha + \nu\theta) \right| < \epsilon/2 \quad \text{for } n > N_1.$$

Quite similarly we can choose N_2 such that

$$(11) \quad \left| \int_0^1 P(x) dx - \frac{1}{n} \sum_{\nu=1}^n P(\alpha + \nu\theta) \right| < \epsilon/2 \quad \text{for } n > N_2.$$

Let us take $N = \max(N_1, N_2)$, then we have

$$(12) \quad \begin{aligned} \int_0^1 p(x) dx - \epsilon/2 &< \frac{1}{n} \sum_{\nu=1}^n p(\alpha + \nu\theta) \leq \frac{n_J(\alpha)}{n} \\ &\leq \frac{1}{n} \sum_{k=1}^n P(\alpha + \nu\theta) < \int_0^1 P(x) dx + \epsilon/2 \end{aligned}$$

from (5), (10), and (11), and finally

$$(13) \quad \int_0^1 f(x) dx - \epsilon \leq \frac{n_J(\alpha)}{n} \leq \int_0^1 f(x) dx + \epsilon$$

from (2) and (12). Since $\int_0^1 f(x) dx = l$ is the length of the interval J , this proves our proposition.

We shall proceed to the proof of Theorem 1. Using Lemma 1, we can choose a positive integer k such that for any integer n at least one of the k numbers $n\xi + \xi, n\xi + 2\xi, \dots, n\xi + k\xi$ is contained with modulus 1 in the interval I . From the assumption on a_n (since we can suppose $c > 1$ without any loss of generality) we have

$$a_{k-m} \geq a_k/c^m \geq a_k/c^k \quad (m \leq k).$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n |\phi(n\xi)| &= \sum_{n=0}^{\infty} \sum_{m=1}^k a_{nk+m} |\phi((nk+m)\xi)| \\ &\geq (d/c^k) \sum_{n=1}^{\infty} a_{nk}. \end{aligned}$$

Consequently

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \sum_{m=1}^k a_{nk+m} \leq \sum_{n=1}^{\infty} \sum_{m=1}^k a_{nk} c^m \\ &\leq \frac{c^{k+1} - 1}{c - 1} \sum_{n=1}^{\infty} a_{nk} \leq \frac{c^k(c^{k+1} - 1)}{d(c - 1)} \sum_{n=1}^{\infty} a_n |\phi(n\xi)| < \infty. \end{aligned}$$

This proves the theorem.

REMARK. This theorem implies Theorem B in §1; in fact if $\phi(x)$ is Riemann integrable, $\phi(x)$ is continuous almost everywhere, therefore the assumption $\int_0^1 |\phi(x)| dx \neq 0$ assures the existence of an interval on which $|\phi(x)|$ is greater than a positive constant.

THEOREM 2. *Besides the assumption on $\phi(x)$ and $\{a_n\}$ in Theorem 1, we assume that $\phi(0) \neq 0$; then the absolute convergence of the series $\sum_{n=1}^{\infty} a_n \phi(nx)$ for any real x_0 implies the convergence of the series $\sum_{n=1}^{\infty} a_n$.*

PROOF. If x_0 is irrational, then the result follows from Theorem 1. In case x_0 is rational, say $x_0 = p/q$ where p and q are integers, the numbers nqx_0 ($n = 1, 2, \dots$) are all equal to 0 with modulus 1. An argument quite similar to that in the proof of Theorem 1 proves also the required result.

This theorem implies Theorem A in §1 for the case of cosine series. Corresponding to the case of sine series we shall prove the following theorem.

THEOREM 3. *Let $\phi(x)$ and a_n satisfy the same conditions as in Theorem 1. If the series $\sum_{n=1}^{\infty} a_n \phi(nx)$ converges absolutely at an irrational x_0 or at a rational x_0 , $x_0 = p/q$ say, where p and q are relatively prime integers and there exists at least one of the numbers r/q ($r = 1, \dots, q-1$) which is not the zero of $\phi(x)$, then the series $\sum_{n=1}^{\infty} a_n$ converges.*

PROOF. For irrational x_0 , this reduces to Theorem 1. Let us suppose that $\phi(r_0/q) \neq 0$, $0 < r_0 \leq q-1$. Since p and q are relatively prime, there are two integers u and v such that $up + vq = 1$, that is, $up/q = 1/q - v$. Consequently there is a number in the ur_0 consecutive numbers of the sequence $\{nx_0\}$ which is equal to r_0/q with modulus 1. Therefore the same argument as in the proof of Theorem 1 gives the required result.

3. Double trigonometrical series. We shall first state the following lemmas which are generalizations of the results of Mazur-Orlicz [1].

LEMMA 2. *If $f(x)$ is measurable, bounded, and periodic, we have for every integrable function $g(x)$ in (a, b)*

$$\lim_{m,n \rightarrow \infty} \int_a^b f(\omega_{mn}x + \theta_{mn})g(x)dx = \mathfrak{M}(f) \int_a^b g(x)dx,$$

where $\mathfrak{M}(f) = \int_0^l f(x)dx/l$, l being the period of $f(x)$, and $\{\omega_{mn}\}$ and $\{\theta_{mn}\}$ are any sequences of real numbers such that $\lim \omega_{mn} = +\infty$.

PROOF. If $m_i, n_i \rightarrow \infty$ ($i \rightarrow \infty$), we have by a result of Mazur and Orlicz [1, Lemma 1]

$$\lim_{i \rightarrow \infty} \int_a^b f(\omega_{m_i n_i}x + \theta_{m_i n_i})g(x)dx = \mathfrak{M}(f) \int_a^b g(x)dx.$$

This proves Lemma 2.

LEMMA 3. *If $f(x)$ is measurable and periodic, then we have almost everywhere,*

$$\limsup_{m,n \rightarrow \infty} |a_{mn} f(\omega_{mn}x + \theta_{mn})| = \limsup_{m,n \rightarrow \infty} |a_{mn}| \operatorname{ess\,sup} |f(x)|,$$

where $\{\omega_{mn}\}$, $\{\theta_{mn}\}$ are the same as in Lemma 2 and $\{a_{mn}\}$ is any sequence of real numbers.

PROOF. We can choose a sequence of suffixes $\{m_i, n_i\}$ such that $m_i, n_i \rightarrow \infty$ ($i \rightarrow \infty$) and $\lim_{i \rightarrow \infty} |a_{m_i n_i}| = \limsup_{m,n \rightarrow \infty} |a_{mn}|$. Then by Mazur and Orlicz [1, Theorem 1] we have almost everywhere

$$\begin{aligned} \limsup_{i \rightarrow \infty} |a_{m_i n_i} f(\omega_{m_i n_i} x + \theta_{m_i n_i})| &= \limsup_{i \rightarrow \infty} |a_{m_i n_i}| \operatorname{ess\,sup} |f(x)| \\ &= \limsup_{m,n \rightarrow \infty} |a_{mn}| \operatorname{ess\,sup} |f(x)|. \end{aligned}$$

It follows that almost everywhere

$$\limsup_{m,n \rightarrow \infty} |a_{mn} f(\omega_{mn}x + \theta_{mn})| \geq \limsup_{m,n \rightarrow \infty} |a_{mn}| \operatorname{ess\,sup} |f(x)|.$$

Since the inverse inequality is evident, our lemma is proved.

LEMMA 4. *If $f(x)$ and $g(x)$ are linearly independent, bounded, measurable, and periodic with the same period, then for any sequences $\{a_{mn}\}$ and $\{b_{mn}\}$, we have almost everywhere*

$$\begin{aligned} \limsup_{m,n \rightarrow \infty} |a_{mn} f(\omega_{mn}x + \theta_{mn}) + b_{mn} g(\omega_{mn}x + \theta_{mn})| \\ \geq c \limsup_{m,n \rightarrow \infty} (|a_{mn}| + |b_{mn}|), \end{aligned}$$

where $c = \inf_{|a|+|b|=1} \{ \operatorname{ess\,sup} |af(x) + bg(x)| \} > 0$. (See Mazur-Orlicz [1, Theorem 3 and its corollary].)

PROOF. We can choose a sequence of indices m_i, n_i ($m_i, n_i \rightarrow \infty$ as $i \rightarrow \infty$) such that $\limsup_{m,n \rightarrow \infty} (|a_{mn}| + |b_{mn}|) = \lim_{i \rightarrow \infty} (|a_{m_i n_i}| + |b_{m_i n_i}|)$. Then we have almost everywhere

$$\begin{aligned} \limsup_{m,n \rightarrow \infty} |a_{mn} f(\omega_{mn}x + \theta_{mn}) + b_{mn} g(\omega_{mn}x + \theta_{mn})| \\ \geq \limsup_{i \rightarrow \infty} |a_{m_i n_i} f(\omega_{m_i n_i} x + \theta_{m_i n_i}) + b_{m_i n_i} g(\omega_{m_i n_i} x + \theta_{m_i n_i})| \\ \geq c \limsup_{i \rightarrow \infty} (|a_{m_i n_i}| + |b_{m_i n_i}|) \\ = c \limsup_{m,n \rightarrow \infty} (|a_{mn}| + |b_{mn}|), \end{aligned}$$

by a result of Mazur and Orlicz [1, Theorem 3 and its corollary]. This proves the lemma.

We shall now prove the following theorem which is a generalization of Cantor-Lebesgue's theorem for double trigonometric series.

THEOREM 4. *Let $f(x)$, $g(x)$ be any measurable, periodic functions, and $\phi(y)$, $\psi(y)$ be linearly independent, bounded, measurable, and periodic functions with the same period, and let $\{\lambda_{mn}\}$, $\{\omega_{mn}\}$, and $\{\omega'_{mn}\}$ be the sequences which tend to ∞ as $m, n \rightarrow \infty$, and $\{\tau_{mn}\}$, $\{\theta_{mn}\}$, $\{\theta'_{mn}\}$ be any sequences. Then for any sequences $\{p_{mn}\}$ and $\{q_{mn}\}$, there exists a constant $c > 0$ depending only on ϕ and ψ , such that*

$$\begin{aligned} \limsup_{m, n \rightarrow \infty} & \left| p_{mn} f(\omega_{mn} x + \theta_{mn}) \phi(\lambda_{mn} y + \tau_{mn}) \right. \\ & \left. + q_{mn} g(\omega'_{mn} x + \theta'_{mn}) \psi(\lambda_{mn} y + \tau_{mn}) \right| \\ & \geq c \max \left\{ \limsup_{m, n \rightarrow \infty} |p_{mn}| \operatorname{ess\,sup} |f(x)|, \right. \\ & \left. \limsup_{m, n \rightarrow \infty} |q_{mn}| \operatorname{ess\,sup} |g(x)| \right\} \end{aligned}$$

for almost every (x, y) .

PROOF. If we fix an x , we have

$$\begin{aligned} \limsup_{m, n \rightarrow \infty} & \left| p_{mn} f(\omega_{mn} x + \theta_{mn}) \phi(\lambda_{mn} y + \tau_{mn}) \right. \\ & \left. + q_{mn} g(\omega'_{mn} x + \theta'_{mn}) \psi(\lambda_{mn} y + \tau_{mn}) \right| \\ & \geq c \limsup_{m, n \rightarrow \infty} \left\{ |p_{mn} f(\omega_{mn} x + \theta_{mn})| + |q_{mn} g(\omega'_{mn} x + \theta'_{mn})| \right\} \\ & \geq c \max \left\{ \limsup_{m, n \rightarrow \infty} |p_{mn} f(\omega_{mn} x + \theta_{mn})|, \right. \\ & \left. \limsup_{m, n \rightarrow \infty} |q_{mn} g(\omega'_{mn} x + \theta'_{mn})| \right\} \end{aligned}$$

by Lemma 4. Applying Lemma 3, we have for almost every x

$$\limsup_{m, n \rightarrow \infty} |p_{mn} f(\omega_{mn} x + \theta_{mn})| = \limsup_{m, n \rightarrow \infty} |p_{mn}| \operatorname{ess\,sup} |f(x)|$$

and

$$\limsup_{m, n \rightarrow \infty} |q_{mn} g(\omega'_{mn} x + \theta'_{mn})| = \limsup_{m, n \rightarrow \infty} |q_{mn}| \operatorname{ess\,sup} |g(x)|.$$

Then by Fubini's theorem we get the theorem.

COROLLARY. *Let*

$$A_{mn}(x, y) = a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny \\ + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \cos ny.$$

If $A_{mn}(x, y)$ tends to zero as $m, n \rightarrow \infty$, for every (x, y) belonging to a plane set of positive measure, then

$$\rho_{mn} = (a_{mn}^2 + b_{mn}^2 + c_{mn}^2 + d_{mn}^2)^{1/2} \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

This is the Reves-Szász generalization of the Cantor-Lebesgue theorem [1, Theorem 1].

PROOF. Let us put in Theorem 4

$$f(x) = g(x) = \cos x, \quad \phi(y) = \cos y, \quad \psi(y) = \sin y, \\ p_{mn}^2 = a_{mn}^2 + b_{mn}^2, \quad q_{mn}^2 = c_{mn}^2 + d_{mn}^2.$$

Then we may write (see Reves and Szász [2, Theorem 1])

$$A_{mn}(x, y) = p_{mn} \cos (mx - \theta_{mn}) \cos ny + q_{mn} \cos (mx - \theta'_{mn}) \sin ny.$$

Therefore we get the corollary by Theorem 4.

THEOREM 5. Let $A_{mn}(x, y)$ and ρ_{mn} be the same as in the above corollary. Then the plane set E ,³

$$E = \left\{ (x, y) \mid \limsup_{m, n \rightarrow \infty} \sum_{i, j=1}^{m, n} |A_{ij}(x, y)| / \sum_{i, j=1}^{m, n} \rho_{ij} < \alpha \right\}$$

is of measure zero, provided that $\alpha < 4/2^{1/2}\pi^2$ and $\sum_{m, n=1}^{\infty} \rho_{mn} = \infty$.

PROOF. Let $f(x, y)$ be the characteristic function of the set E , then we have

$$\alpha f(x, y) \geq \limsup_{m, n \rightarrow \infty} f(x, y) \sum_{i, j=1}^{m, n} |A_{ij}(x, y)| / \sum_{i, j=1}^{m, n} \rho_{ij} \\ = \limsup_{m, n \rightarrow \infty} \frac{f(x, y) \sum_{i, j=1}^{m, n} |A_{ij}(x, y)|}{\sum_{i, j=1}^{m, n} (\Delta_{ij}(x))^{1/2}} \frac{\sum_{i, j=1}^{m, n} (\Delta_{ij}(x))^{1/2}}{\sum_{i, j=1}^{m, n} \rho_{ij}} \\ \geq \limsup_{m, n \rightarrow \infty} \frac{1}{2^{1/2}} \frac{f(x, y) \sum_{i, j=1}^{m, n} |A_{ij}(x, y)|}{\sum_{i, j=1}^{m, n} (\Delta_{ij}(x))^{1/2}} \\ \cdot \frac{\sum_{i, j=1}^{m, n} \{ |p_{ij} \cos (ix - \theta_{ij})| + |q_{ij} \cos (ix - \theta'_{ij})| \}}{\sum_{i, j=1}^{m, n} \rho_{ij}}$$

³ Of course, we suppose that E is contained in the square $(0, 2\pi; 0, 2\pi)$.

where $\Delta_{ij}(x) = p_{ij}^2 \cos^2 (ix - \theta_{ij}) + q_{ij}^2 \cos^2 (ix - \theta'_{ij})$.

Applying Lebesgue's theorem we get

$$\alpha \int_0^{2\pi} f(x, y) dy \geq \frac{1}{2^{1/2}} \limsup_{m, n \rightarrow \infty} \frac{\sum_{i, j=1}^{m, n} \int_0^{2\pi} f(x, y) |A_{ij}(x, y)| dy}{\sum_{i, j=1}^{m, n} (\Delta_{ij}(x))^{1/2}} \cdot \frac{\sum_{i, j=1}^{m, n} \{ |p_{ij} \cos (ix - \theta_{ij})| + |q_{ij} \cos (ix - \theta'_{ij})| \}}{\sum_{i, j=1}^{m, n} \rho_{ij}}.$$

Since $\sum \rho_{mn} = \infty$, we can easily verify that

$$\sum_{i, j=1}^{\infty} (\Delta_{ij}(x))^{1/2} \geq \frac{1}{2^{1/2}} \sum_{i, j=1}^{\infty} \{ |p_{ij} \cos (ix - \theta_{ij})| + |q_{ij} \cos (ix - \theta'_{ij})| \} = \infty$$

for almost every x . For such x , we have by Lemma 2

$$\lim_{m, n \rightarrow \infty} \frac{\sum_{i, j=1}^{m, n} \int_0^{2\pi} f(x, y) |A_{ij}(x, y)| dy}{\sum_{i, j=1}^{m, n} (\Delta_{ij}(x))^{1/2}} = \frac{2}{\pi} \int_0^{2\pi} f(x, y) dy.$$

Hence

$$\alpha \int_0^{2\pi} f(x, y) dy \geq \frac{2}{2^{1/2}\pi} \int_0^{2\pi} f(x, y) dy \cdot \limsup_{m, n \rightarrow \infty} \frac{\sum_{i, j=1}^{m, n} \{ |p_{ij} \cos (ix - \theta_{ij})| + |q_{ij} \cos (ix - \theta'_{ij})| \}}{\sum_{i, j=1}^{m, n} \rho_{ij}}.$$

Since by Lemma 2

$$\begin{aligned} \lim_{i, j \rightarrow \infty} \int_0^{2\pi} \int_0^{2\pi} f(x, y) |\cos (ix - \theta_{ij})| dx dy \\ = \frac{2}{\pi} \int_0^{2\pi} \int_0^{2\pi} f(x, y) dx dy, \end{aligned}$$

we have

$$\alpha \int_0^{2\pi} \int_0^{2\pi} f(x, y) dx dy \geq \frac{4}{2^{1/2}\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) dx dy.$$

If $\alpha < 4/2^{1/2}\pi^2$, then we get

$$\int_0^{2\pi} \int_0^{2\pi} f(x, y) dx dy = 0,$$

that is, E is of measure zero. Thus we get the theorem.

This is a generalization of a theorem of Salem [3, Theorem X]. Denjoy-Lusin's theorem for double trigonometrical series, which was given by Reves and Szász [2, Theorem 2], can be easily derived from Theorem 4.

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