A REMARK ON CHAIN-HOMOTOPY

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1. **Introduction.** Let X, Y be finite abstract complexes in the sense of [1] (numbers in square brackets refer to the bibliography at the end of this note). Integral coefficients will be used throughout in dealing with the homology groups attached to complexes. Two chain-mappings τ_1, τ_2 are termed chain-homotopic, in symbols $\tau_1 \simeq \tau_2$, if there exists, for each dimension p, a homomorphism

$$D: \quad C_p(X) \to C_{p+1}(Y)$$

such that

$$\partial D c_p^X + D \partial c_p^X = \tau_2 c_p^X - \tau_1 c_p^X$$

for every p-chain c_p^X of the complex X. If $\tau_1 \simeq \tau_2$, then $\tau_1 \sim \tau_2$ (that is, τ_1, τ_2 induce identical homomorphisms from the homology groups of X into the homology groups of Y). It is well known that the converse is generally false. Several examples to this effect may be found in [2]. A very simple example is given in §2 below. On the other hand, if X is torsion-free, then the relation $\tau_1 \sim \tau_2$ does imply the relation $\tau_1 \simeq \tau_2$ (see [1, pp. 155–157]). The purpose of this note is to further clarify the role of torsion in this connection, by proving the following result concerning simplicial complexes.

THEOREM. *If the finite simplicial complex K is not torsion-free, then there exists a finite simplicial complex L and a pair of chain-mappings τ_1, τ_2: $K \to L$, such that τ_1, τ_2 are homologous but not chain-homotopic.*

2. **Preliminary remarks on finite abstract complexes.** Even though our theorem is concerned with the simplicial case, clarity will be gained by developing part of the argument for finite abstract complexes. Let us denote by $C_p(X)$, $Z_p(X)$, $B_p(X)$, $H_p(X)$ the groups of integral chains, cycles, boundaries, and the homology groups of the finite abstract complex X (see [1]). The following points should be remembered in working with these concepts.

(a) If a certain dimension p is missing, then the group $C_p(X)$ is defined as consisting of a zero-element alone. As a consequence, every $(p+1)$-chain is then a cycle, and the only $(p-1)$-boundary is the 0-element of $C_{p-1}(X)$.

(b) The incidence-numbers $[x_p : x_{p-1}]$ are not required to have only

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the values 0, $+1$, -1, although this happens to be the case in many important special instances.

Now let us consider the following example (abstracted from the case of the projective plane). Let X consist of a single 1-cell x_1 and a single 0-cell x_0, with the incidence-number $[x_1:x_0]=2$. Let Y consist of a single 1-cell y_1 and a single 0-cell y_0, with the incidence-number $[y_1:y_0]=0$. The following statements are readily verified.

(i) X and Y are abstract complexes in the sense of [1], with the boundary relations $\partial x_1=2x_0$, $\partial y_1=0$.

(ii) On defining $\tau: X\to Y$, $\sigma:X\to Y$ by the relations

$$\tau x_1 = y_1, \qquad \tau x_0 = 0, \qquad \sigma x_1 = 0, \qquad \sigma x_0 = 0,$$

both τ and σ are chain-mappings, and $\tau\sim\sigma$.

(iii) On the other hand, the relation $\tau\simeq\sigma$ does not hold. Indeed, assume that $\tau\simeq\sigma$. Then there would exist a homotopy operator $D:C_p(X)\to C_{p+1}(Y)$ satisfying the relation $\partial Dc_p^X+D\partial_p^X-\tau c_p^X=0$. In particular, we would have

$$\partial Dx_1 + D\partial x_1 - \tau x_1 = 0.$$

Now $Dx_1=0$, since Y has no 2-cells, and $D\partial x_1=2Dx_0$. Here Dx_0 is a 1-cell of Y, hence $Dx^0=ky_1$, where k is an integer. There follows the relation $(2k-1)y_1=0$, a contradiction, since the integer $2k-1$ is odd and hence different from zero.

This example shows that homologous chain-mappings need not be homotopic (as stated in the introduction, examples to this effect were constructed already in [2]). In constructing the above example, we took advantage of the fact that the incidence-numbers in a finite abstract complex are not restricted to the values 0, $+1$, -1. However, as the theorem stated in the introduction shows, the same phenomenon arises in the case of simplicial complexes where the incidence-numbers are restricted to the values 0, $+1$, -1.

3. **A special chain-mapping for finite abstract complexes.** Let X be a finite abstract complex. We shall set up presently a special chain-mapping $\tau:X\to X$. For clarity, we shall use τ_p to refer to τ acting in dimension p. If a certain dimension p is missing, then according to the conventions stated above the group $C_p(X)$ consists of a zero-element alone, and then τ_p is the trivial homomorphism carrying zero into zero. Accordingly, τ_p is thought of as acting in all dimensions $-\infty<p<+\infty$. We assert that there exists a set of homomorphisms $\tau_p:C_p(X)\to C_p(X)$, such that the following conditions hold.

(i) $\partial\tau_p c_p=\tau_{p-1}\partial c_p$, $-\infty<p<\infty$.

(ii) If $c_p \in B_p(X)$, then $\tau_p c_p = c_p$, $-\infty < p < \infty$.

(iii) If X is torsion-free in a certain dimension p, then $\tau_p z_p \in B_p(X)$ for every cycle z_p in that dimension.

(iv) If X does have torsion in a certain dimension p, then $\tau_p c_p = c_p$ for every p-chain c_p in that dimension.

In the preceding statements, $B_p(X)$ denotes the group of p-boundaries in X. The existence of τ_p will be proved by induction.

Part 1. Let p be an integer such that τ_q can be defined in all dimensions $q<p$ as required. Then τ_p can also be defined as required, without changing the homomorphisms τ_q in the dimensions $q<p$. We divide the proof of this assertion into two sections.

Case I. X does have torsion in dimension p. In view of condition (iv), we have then no choice but to set $\tau_p c_p = c_p$, for every p-chain c_p of X. Then (iv) holds, (ii) is obvious, (iii) is vacuous, and (i) is verified as follows. We have now $\partial \tau_p c_p = \partial c_p$. On the other hand, since $\partial c_p \in B_{p-1}$, the induction assumption (ii) applies with $q=p-1$, yielding $\tau_{p-1}\partial c_p = \partial c_p$. Thus $\partial \tau_p c_p = \tau_{p-1}\partial c_p = \partial c_p$.

Case II. X is torsion-free in dimension p. Taking first the general case when the groups $C_p(X), Z_p(X), B_p(X)$ do not reduce to the zero-element alone, we choose a canonical base for $C_p(X)$ (see [1, pp. 102–103]). As a matter of fact, for our present purposes it is sufficient to observe that since X is torsion-free in the dimension p under consideration, we can choose for $C_p(X)$ a base consisting of three aggregates b_p^i, z_p^j, c_p^k of p-chains such that the b_p^i form a base for $B_p(X)$, the b_p^i and the z_p^j form a base for $Z_p(X)$, and of course the b_p^i, z_p^j, c_p^k form a base for $C_p(X)$. The existence of such a base is due of course to the fact that by assumption the p-dimensional torsion-coefficients all vanish. Now consider ∂c_p^k. This being a boundary in dimension $p-1$, by the induction assumption (ii), applied with $q=p-1<p$, we have

$$\tau_{p-1}\partial \overset{k}{c}_p = \partial \overset{k}{c}_p. \tag{1}$$

We now define $\tau_p: C_p(X) \to C_p(X)$ by assigning the images under τ_p of b_p^i, z_p^j, c_p^k as follows:

$$\tau_p \overset{i}{b}_p = \overset{i}{b}_p, \qquad \tau_p \overset{j}{z}_p = 0, \qquad \tau_p \overset{k}{c}_p = \overset{k}{c}_p. \tag{2}$$

We have to check the conditions (i)–(iv). The conditions (ii), (iii) are obviously satisfied in view of (2), condition (iv) is now vacuous, and condition (i) is verified as follows. Clearly, we have to check only that (i) holds for each of the b_p^i, z_p^j, c_p^k. Now since b_p^i is a boundary, we have $\partial b_p^i = 0$ and hence $\tau_{p-1}\, \partial b_p^i = 0$, and by (2) also $\partial \tau_p b_p^i = \partial b_p^i = 0$.

Similarly, $\partial z_p^j = 0$, hence $\tau_{p-1}\partial z_p^j = 0$, and by (2) also $\partial \tau_p z_p^i = \partial 0 = 0$. Finally, by (1) and (2), $\partial \tau_p c_p^k = \partial c_p^k = \tau_{p-1}\partial c_p^k$.

Thus Case II is disposed of in the general case. If the b_p^i are missing, or if both the b_p^i and the z_p^j are missing, and so forth, the same argument applies with trivial modifications.

Part II. Since X is finite, we can choose an integer p_0 such that all dimensions $q < p_0$ are missing. For $q < p_0$, $C_q(X)$ reduces then to a zero-element, and we define of course τ_q as the trivial homomorphism $\tau_q 0 = 0$. The conditions (i)–(iv) manifestly hold then for all $q < p_0$. By Part I, we can therefore extend the definition of τ, subject to the conditions (i)–(iv), successively to the dimensions p_0, p_0+1, $\cdots$. As a matter of fact, since X is finite, we arrive after a finite number of steps at a dimension q such that $C_p(X)$ reduces to a zero-element for all $p > q$. For all $p > q$, the trivial homomorphism $\tau_p 0 = 0$ satisfies then the conditions (i)–(iv). Thus our induction process is actually finite.

4. **Preliminary remarks on simplicial complexes.** Let K be a nonempty finite simplicial complex. We take an auxiliary vertex v not occurring in K, and arrange the vertices of K in an arbitrary order $v_1, \cdots, v_m$. We then agree to use in the sequel the arrangement

$$v, v_1, \cdots, v_m \tag{3}$$

for purposes of orienting simplexes. In particular, the groups $C_p(K)$, $Z_p(K)$, $B_p(K)$, $H_p(K)$ will be set up in terms of the particular ordering (3). Now let S be a nonempty subcomplex of K, and let Γ be the cone over S with vertex v. The simplexes of Γ are oriented in terms of the ordering (3). We consider next the simplicial complex $L = K \cup \Gamma$, and we denote by $\eta: K \to L$ the injection from K into L and by $\lambda: K \to L$ the zero-homomorphism. That is, λ carries every element of $C_p(K)$ into the zero-element of $C_p(L)$, for every p. All these simplicial complexes are taken unaugmented. Accordingly, all negative dimensions are missing, and every 0-chain is a cycle. Furthermore, all these simplicial complexes are subject to the conventions stated above for abstract complexes. The following well known facts will be used.

(a) If a cycle z_p^k of K lies in S, then z_p^k bounds in Γ, provided that $p \geqq 1$.

(b) If β is the dimension of S, and if $c_{\beta+2}^L$ is a $(\beta+2)$-chain of L, then we have a $(\beta+2)$-chain $d_{\beta+2}^k$ of K such that $\eta d_{\beta+2}^k = c_{\beta+2}^L$. Indeed, since the dimension of the cone Γ over S is $\beta+1$, all the simplexes of $c_{\beta+2}^L$ occur in K (as a matter of fact, $d_{\beta+2}^k$ is merely $c_{\beta+2}^L$ considered as a chain in K).

5. **Proof of the theorem stated in the introduction.** Let K be a finite simplicial complex which is *not torsion-free*, and let β be the highest dimension in which torsion occurs. Let S consist of all the simplexes of K with dimension less than or equal to β, and let Γ be the cone over S with vertex v, where the conventions stated in §4 are used. Let $\tau: K \to K$ be the special chain-mapping described in §3. We denote, as in §4, by η the injection from K into L, and by λ the zero homomorphism from K into L. We have then the chain-mappings

$$\sigma_1 = \eta\tau: K \to L, \qquad \sigma_2 = \lambda\tau: K \to L.$$

We assert that σ_1 and σ_2 are homologous but not chain-homotopic. Since σ_2 is the zero chain-mapping, we have to check first that

$$\sigma_1 \overset{k}{z}_p = \eta\tau \overset{k}{z}_p \in B_p(L)$$

for every cycle z_p^k of K. Now if K is torsion-free in a particular dimension p, then $\tau z_p^k \in B_p(K)$, by property (iii) of τ, hence a fortiori $\sigma_1 z_p^k = \eta\tau z_p^k \in B_p(L)$. If K is not torsion-free in dimension p, then $\beta \geqq p$ by the definition of β, and $p \geqq 1$, since a simplicial complex is torsion-free in dimension zero. Accordingly, z_p^k lies in the subcomplex S, and bounds in the cone Γ and hence also in L. Thus $\eta z_p^k \in B_p(L)$. On the other hand, since K is not torsion-free in dimension p, we have $\tau z_p^k = z_p^k$ by property (iv) of τ. It follows that $\sigma_1 z_p^k = \eta\tau z_p^k = \eta z_p^k \in B_p(L)$.

We shall show presently that the relation $\sigma_1 \simeq \sigma_2$ fails to hold. Indeed, assume that there exists a homotopy operator

$$D: C_p(K) \to C_{p+1}(L)$$

for the pair σ_1, σ_2. Since σ_2 is the zero-homomorphism, we would then have identically

$$\partial D \overset{k}{c}_p + D\partial \overset{k}{c}_p - \eta\tau \overset{k}{c}_p = 0, \tag{4}$$

for all chains c_p^k of K. A contradiction is now readily obtained in the following manner. Since by assumption K has torsion in dimension β, we have a particular cycle z_β^k and a nonzero integer n, such that nz_β^k bounds in K:

$$n \overset{k}{z}_\beta = \partial \overset{k}{c}_{\beta+1}, \qquad n \neq 0, \tag{5}$$

while on the other hand

$$\overset{k}{z}_\beta \text{ does not bound in } K. \tag{6}$$

We have then by (4) the relation

$$\partial D\overset{k}{c}_{\beta+1} + D\partial\overset{k}{c}_{\beta+1} - \eta\tau\overset{k}{c}_{\beta+1} = 0. \tag{7}$$

Now $Dc^k_{\beta+1}$ is a $(\beta+2)$-chain of L. Hence, by the remark (b) in §5, we can write

$$D\overset{k}{c}_{\beta+1} = \eta\overset{k}{d}_{\beta+2}. \tag{8}$$

(5), (7) yield (since η commutes with ∂):

$$\eta\partial\overset{k}{d}_{\beta+2} + nD\overset{k}{z}_{\beta} - \eta\tau\overset{k}{c}_{\beta+1} = 0. \tag{9}$$

Denoting by $\pi: L \to K$ the projection from L onto K, and applying π on the left in (9), we get

$$\partial\overset{k}{d}_{\beta+2} + n\pi D\overset{k}{z}_{\beta} - \tau\overset{k}{c}_{\beta+1} = 0. \tag{10}$$

Now $\pi Dz^k_\beta = e^k_{\beta+1}$, a $(\beta+1)$-chain in K. Making this substitution in (10), and applying the boundary operator ∂, we obtain

$$n\partial\overset{k}{e}_{\beta+1} - \partial\tau\overset{k}{c}_{\beta+1} = 0. \tag{11}$$

Now $\partial\tau c^k_{\beta+1} = \tau\partial c^k_{\beta+1} = n\tau z^k_\beta$ by (5). Since K does have torsion in dimension β, we have $\tau z^k_\beta = z^k_\beta$ by property (iv) of τ. Thus (11) yields

$$n\partial\overset{k}{e}_{\beta+1} - n\overset{k}{z}_{\beta} = 0. \tag{12}$$

Since $n \neq 0$, it follows that $z^k_\beta = \partial e^k_{\beta+1}$. Thus z^k_β would bound in K, in contradiction with (6).

Bibliography

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