THE PROBABILITY OF A CORRECT RESULT WITH A CERTAIN ROUNDING-OFF PROCEDURE

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1. **Introduction.** In this paper we are concerned with the addition of n arbitrary numbers expressed with the base B of numeration. We describe a procedure for obtaining this sum to a prescribed number of places, and determine the probability that the result is correct, under the assumption that the numbers to be added are uniformly distributed. We also derive asymptotic formulas for the probability in the case where n becomes infinite, and where B becomes infinite.

2. **Description of the procedure.** Let the sum of n numbers, known to an arbitrary accuracy, be required only to the nearest unit in a prescribed place. Surely the best way of obtaining this is to add the numbers directly, and round off the sum to the desired place. This may well be inconvenient or impossible. Instead let each addend be rounded off to one more place than required in the final result. Let these approximate addends be added, and their sum rounded off one more place. This process does not always give the correct result as is shown by the addition of 3.14, 4.14, and 5.24 with accuracy to the nearest integer required. The correct result is 13, while the approximate process gives 12.

The procedure described includes the case of rounding off the addends to more than one extra place. Rounding to k extra places is exactly equivalent to rounding to one extra place with the base of numeration B^k.

3. **Probability of a correct result.**

THEOREM 1. *Let $x_1, x_2, \cdots, x_n$ be uniformly distributed real variables expressed in the base B of numeration. Let $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ be the nearest-integer approximations to $x_1, \cdots, x_n$. Let S and $\overline{S}$ be the values obtained when the sums*

$$\sum_{j=1}^{n} x_j \quad \textit{and} \quad \sum_{j=1}^{n} \bar{x}_j$$

respectively are rounded off to the nearest multiple of the base B.

Then the probability that $S=\overline{S}$ is

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$$P_n^{(1)} = \frac{2}{\pi B}\int_0^\infty \left(\frac{\sin t}{t}\right)^{n-1} \frac{\sin^2 Bt}{t^2}\,dt \tag{3.1}$$

if B is an odd integer, and is

$$P_n^{(2)} = \frac{2}{\pi B}\int_0^\infty \left(\frac{\sin t}{t}\right)^{n-1} \frac{\sin^2 Bt}{t^2}\cos t\,dt \tag{3.2}$$

if B is an even integer.

REMARKS. The above theorem is stated for the case that the final result is desired to the nearest integer in the place two to the left of the point. All other places are included, however, since they merely correspond to a change of scale in the variables which does not alter the probability.

PROOF OF THEOREM 1. We consider $(x_1, \cdots, x_n)$ as coordinates in Euclidean n-space. We shall be particularly concerned with n-cubes of side one unit, with edges parallel to the axes, and centered at points with integral coordinates. In such an n-cube, hereafter called a cube, the sum $\sum \bar{x}_j$ is constant. The sum $\sum x_j$ will not be constant in such a cube. With each cube is associated the residue-class (modulo B) to which the sum $\sum \bar{x}_j$ within it belongs. The measure of the portion of each such cube for which $S=\overline{S}$ will depend on the residue class associated with the cube, and on no other property of it. This is because $S-\overline{S}$ is a periodic function of the x's with period B. The B types of cubes are equally distributed in space. Therefore to determine the probability that $S=\overline{S}$ we consider a set of B cubes, one associated with each residue class, and compute the probability that $S=\overline{S}$ for this set.

If B is an odd integer, we select as our set of B cubes the portion of space

$$-1/2 \leqq x_j \leqq 1/2, \quad j = 1, \cdots, n-1, \quad -B/2 \leqq x_n \leqq B/2.$$

For each cube, $\overline{S}=0$, so we seek the probability that $S=0$ within them. That is that

$$-B/2 \leqq \sum x_j \leqq B/2.$$

We introduce the distribution functions $F_j(x)$ for the x's and their associated characteristic functions

$$\phi_j(t) = \int_{-\infty}^{\infty} e^{itx} dF_j(x).$$

The characteristic function for the sum $\sum x_j$ will be the product

$$\phi(t) = \phi_1(t) \cdots \phi_n(t),$$

and the distribution function will be determined to within a constant by

$$F(b) - F(a) = \lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{-ibt} - e^{-iat}}{-it} \phi(t)dt.$$

We find that for $j=1, 2, \cdots, n-1$

$$F_j(x) = \begin{cases} 0, & x \leqq -1/2, \\ x + 1/2, & -1/2 \leqq x \leqq 1/2, \\ 1, & x \geqq 1/2, \end{cases} \qquad \phi_j(t) = \frac{2}{t} \sin \frac{t}{2},$$

while

$$F_n(x) = \begin{cases} 0, & x \leqq -B/2, \\ x/B + 1/2, & -B/2 \leqq x \leqq B/2, \\ 1, & x \geqq B/2, \end{cases} \qquad \phi_n(t) = \frac{2}{Bt} \sin \frac{Bt}{2} \cdot$$

Thus

$$\phi(t) = \frac{2^n}{Bt^n} \sin^{n-1} \frac{t}{2} \sin \frac{Bt}{2},$$

and the required probability is

$$P_n^{(1)} = F(B/2) - F(-B/2) = \lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{2}{t} \sin \frac{Bt}{2} \phi(t)dt.$$

Making the change of variable $t=2u$ we obtain

$$P_n^{(1)} = \lim_{A\to\infty} \frac{1}{\pi B} \int_{-A}^{A} \left(\frac{\sin u}{u}\right)^{n-1} \frac{\sin^2 Bu}{u^2} du.$$

Since the integrand is an even function, this reduces to (3.1).

The case for which B is an even integer is different because of the question of rounding off when the last digit is $B/2$. It is customary to form $\overline{S}$ so that it equals an even multiple of B if $\sum \bar{x}_j$ is congruent to $B/2$. For our purposes it is simpler to make $\overline{S}$ equal the next higher multiple of B in such a case. The probability of correct result is the same with either convention.

We choose for the B cubes the portion of space

$$-1/2 \leqq x_j \leqq 1/2, j = 1, \cdots, n-1, \quad -B/2 - 1/2 \leqq x_n \leqq B/2 - 1/2.$$

We have again that $\overline{S}=0$ in each of the B cubes, so we seek the probability that $S=0$, or that

$$-B/2 \leqq \sum x_j \leqq B/2.$$

The characteristic functions and distribution functions for the x_j are the same as before except that

$$F_n(x) = \begin{cases} 0, & x \leqq -B/2 - 1/2, \\ x/B + 1/2 + 1/2B, & -B/2 - 1/2 \leqq x \leqq B/2 - 1/2, \\ 1, & x \leqq B/2 - 1/2. \end{cases}$$

$$\phi_n(t) = e^{-it/2} \frac{2}{Bt} \sin \frac{Bt}{2} \cdot$$

The other calculations are as before, and we obtain

$$P_n^{(2)} = \frac{1}{\pi B} \int_{-\infty}^{\infty} e^{-iu} \left(\frac{\sin u}{u} \right)^{n-1} \frac{\sin^2 Bu}{u^2} du.$$

Since the integration is over a symmetric range, only the real part of the integrand contributes, and the integral reduces to (3.2).

4. Asymptotic behavior as $n \rightarrow \infty$.

THEOREM 2. *If P_n is either of the probabilities found in Theorem* 1, *we have*

$$\lim_{n\rightarrow\infty} n^{1/2} P_n = B(6/\pi)^{1/2}, \tag{4.1}$$

so that as $n \rightarrow \infty$,

$$P_n \sim B(6/\pi n)^{1/2}. \tag{4.2}$$

PROOF. Let $K_n(t)$ represent the integrand in either (3.1) or (3.2). If δ is any number satisfying $0 < \delta < \pi$,

$$\lim_{n\rightarrow\infty} \int_{\delta}^{\infty} n^{1/2} K_n(t) dt = 0. \tag{4.3}$$

We verify (4.3) by considering the integrals from δ to π, and from π to ∞. The former tends to zero because the maximum of its integrand is less than $B^2 n^{1/2} ((\sin \delta)/\delta)^{n-1}$. The latter tends to zero because its integrand is dominated by $n^{1/2} t^{-n-1}$.

As a consequence of (4.3) the limiting behavior of $n^{1/2} P_n$ is exactly the same as the limiting behavior when the integration in the formula for P_n is over the range from 0 to δ. Also the behavior in this latter case is independent of δ. Thus the asymptotic behavior depends only on the values of $K_n(t)$ in an arbitrarily small neighborhood of $t=0$.

By consideration of Maclaurin series, it can be verified that for any sufficiently small positive ϵ there exists a number δ $(0<\delta<\pi/2)$ for which

$$e^{-t^2/(6-\epsilon)} \leqq \frac{\sin t}{t} \leqq e^{-t^2/6} \tag{4.4}$$

for $0\leqq t\leqq\delta$. We carry through the details only for the case that B is odd. The details for B even involve merely the introduction of cosine factors at appropriate places and do not alter the result. In view of (4.4)

$$\begin{aligned}\frac{2\sin^2 B\delta}{\pi B\delta^2}\left(\frac{(6-\epsilon)n}{n-1}\right)^{1/2}&\int_0^{\delta((n-1)/(6-\epsilon))^{1/2}} e^{-u^2}du \\ &= \frac{2n^{1/2}}{\pi B}\frac{\sin^2 B\delta}{\delta^2}\int_0^{\delta} e^{-(n-1)t^2/(6-\epsilon)}dt \\ &\leqq \frac{2n^{1/2}}{\pi B}\int_0^{\delta}\left(\frac{\sin t}{t}\right)^{n-1}\frac{\sin^2 Bt}{t^2}dt \\ &\leqq \frac{2Bn^{1/2}}{\pi}\int_0^{\delta} e^{-(n-1)t^2/6}dt \\ &= \frac{2B}{\pi}\left(\frac{6n}{n-1}\right)^{1/2}\int_0^{\delta((n-1)/6)^{1/2}} e^{-u^2}du.\end{aligned} \tag{4.5}$$

As a result of (4.5) the upper and lower limits as $n\rightarrow\infty$ of the middle term lie between

$$\frac{\sin^2 B\delta}{B\delta^2}\left(\frac{6-\epsilon}{\pi}\right)^{1/2} \quad \text{and} \quad B\left(\frac{6}{\pi}\right)^{1/2}.$$

The above is true for arbitrarily small δ and ϵ, so that a limit is actually approached. (4.1) and (4.2) follow from the remarks following (4.3).

5. Asymptotic behavior as $B\rightarrow\infty$.

THEOREM 3.

1. *If B is odd, and $B\geqq(n-1)/2$, formula* (3.1) *has the form*

$$P_n^{(1)} = 1 - \frac{c_n}{B}, \tag{5.1}$$

where

$$c_n = \frac{1}{\pi}\int_0^\infty \frac{t^{n-1} - \sin^{n-1} t}{t^{n+1}}\,dt. \tag{5.2}$$

2. *If B is even, and $B \geqq n/2$, formula* (3.2) *has the form*

$$P_n^{(2)} = 1 - \frac{d_n}{B}, \tag{5.3}$$

where

$$d_n = \frac{1}{\pi}\int_0^\infty \frac{t^{n-1} - \sin^{n-1} t}{t^{n+1}} \cos t\,dt. \tag{5.4}$$

3. *As n becomes infinite, both c_n and d_n have the asymptotic formula*

$$(n/6\pi)^{1/2}. \tag{5.5}$$

PROOF. We consider in detail only the case of odd B. For fixed n, the integral in (3.1) considered as a function of B is found to be a polynomial in B which changes its analytical form at unit intervals of B with the final change at $B=(n-1)/2$. To see this we expand the numerator of the integrand of (3.1) as a sum of first powers of sines or cosines. We have the cosines (sines) of $(2B-n+1)t$, $(2B-n+3)t$, and so on, if n is odd (even). We integrate by parts n times, in the course of which the numerator is differentiated n times and the power of t in the denominator is reduced to one. The numerator is then a sum of the sines of the above quantities, each multiplied by a polynomial in B. The asserted form of P_n then follows from

$$\int_0^\infty \frac{\sin Vt}{t}\,dt = \frac{\pi}{2}\operatorname{sgn} V,$$

the changes in analytical form being due to the discontinuity of this integral at $v=0$.

The proof of Theorem 1 shows that as B becomes infinite, the probability (3.1) must approach unity. Therefore for $B \geqq (n-1)/2$ the polynomial in B must be linear, and (3.1) must assume the form (5.1). No higher degree in B would be allowable since a probability is always less than one, and is thus bounded.

The exact form (5.2) for c_n is found by determining the constant term of the polynomial in B for $B \geqq (n-1)/2$. This involves a somewhat extended calculation with trigonometric identities and integrations by parts. The asymptotic formula (5.5) is found by a method analogous to that used in Theorem 2.

Exactly the same reasoning is used to derive the results for even

B. In this case the last change of form of the polynomial occurs at $B=n/2$. The calculations are parallel with those for odd B.

6. **Conclusion.** For $n=1$ or $n=2$ the probabilities (3.1) and (3.2) can be easily evaluated. For B odd we have $P_1=1$, and $P_2=1-1/4B$. If B is even, we have the interesting result that P_1 is *not* equal to 1. In fact $P_1=P_2=1-1/2B$.

Using the results of Theorem 3 together with the remarks of §2, we may give an answer to the question of how many extra places must be retained to insure a high probability of correct result for a given number of addends. From the remark in §2, retaining k extra places with base B is the same as retaining one extra place with base B^k. Thus to determine probability we replace B by B^k in our results. From Theorem 3 if n is large, and k is chosen so that $B^k=n/2$, the probability of correct result is asymptotically

$$P \sim 1 - (2/3\pi n)^{1/2},$$

while if k is larger than this special value, the probability will be still closer to unity. For example if $n=2000$ with $B=10$, using the asymptotic formula we find for $k=3$, $P=.99$, and for $k=4$, $P=.999$.

References

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