ON PERMANENT VECTOR-LINES IN N DIMENSIONS

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Prim and Truesdell have recently given a simple vector proof of Zorawski's condition for the permanence of vector-lines in a moving fluid.[1] Their argument is necessarily by its form confined to three-dimensional space, and it seems of interest to investigate the problem in a Euclidean space of N dimensions, using a slightly different approach.

We consider a Euclidean space E_N with rectangular cartesians x_i. Two vector fields are given, $v_i(x, t)$, $c_i(x, t)$, where t is a parameter (the time); we shall call v_i the *primary* and c_i the *secondary* vector fields; v_i plays the part of *velocity*.

Any vector field defines for given t a congruence of curves in an obvious way, the direction of the curve at each point coinciding with that of the vector field. We are not particularly interested here in the congruence defined by the primary vector field (stream-lines), and we shall use the expression *vector-lines* to refer exclusively to the congruence defined by the secondary field, that is, those curves which satisfy

$$c_i dx_j - c_j dx_i = 0. \tag{1}$$

We use the primary field to generate an infinitesimal transformation $dx_i = v_i(x, t)dt$, the history of an individual particle being obtained by integrating these equations. Any curve at $t = t_0$ will thus generate as its history a succession of curves formed always of the same particles; this succession of curves may be expressed by

$$x_i = f_i(\theta, t) \tag{2}$$

where θ is a parameter which remains constant as we follow a particle. We have then the equations

$$\partial f_i/\partial t = v_i, \qquad \partial f_i/\partial \theta = \lambda_i, \tag{3}$$

where λ_i is a vector tangent to the instantaneous position of the moving curve.

If we follow the history of a curve C which moves with the fluid in the above sense and which is a vector-line at $t = t_0$, it will not in general remain a vector-line. To investigate the conditions under which

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[1] R. Prim and C. Truesdell, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 32–34.

it does so remain, we define a skew-symmetric tensor by

$$\Omega_{ij} = c_i\lambda_j - c_j\lambda_i. \tag{4}$$

Then $\Omega_{ij}=0$ is a necessary and sufficient condition that C should be a vector-line.

Now Ω_{ij} is a function of θ and t, and if we write

$$c_i(x, t) = g_i(\theta, t) \quad \text{when} \quad x_i = f_i(\theta, t), \tag{5}$$

we have

$$\Omega_{ij}(\theta, t) = g_i\partial f_j/\partial\theta - g_j\partial f_i/\partial\theta. \tag{6}$$

We proceed to examine how Ω_{ij} changes as we move with the fluid, its rate of change being $\partial\Omega_{ij}/\partial t$.

For arbitrary variations in the 2-space (2) we have

$$\begin{aligned} \delta c_i &= c_{i,k}\delta x_k + (\partial c_i/\partial t)\delta t \\ &= c_{i,k}(\lambda_k\delta\theta + v_k\delta t) + (\partial c_i/\partial t)\delta t \\ &= (\partial g_i/\partial\theta)\delta\theta + (\partial g_i/\partial t)\delta t, \end{aligned} \tag{7}$$

where the comma denotes partial differentiation and the summation convention is used. Hence

$$\partial g_i/\partial\theta = c_{i,k}\lambda_k, \qquad \partial g_i/\partial t = c_{i,k}v_k + \partial c_i/\partial t. \tag{8}$$

Thus,

$$\frac{\partial}{\partial t}(g_i\partial f_j/\partial\theta) = (c_{i,k}v_k + \partial c_i/\partial t)\lambda_j + c_iv_{j,k}\lambda_k, \tag{9}$$

since

$$\frac{\partial}{\partial t}\left(\frac{\partial f_j}{\partial\theta}\right) = \frac{\partial}{\partial\theta}v_j = v_{j,k}\lambda_k. \tag{10}$$

Accordingly we obtain from (6)

$$\frac{\partial}{\partial t}\Omega_{ij} = \lambda_q(A_{qij} - A_{qji}), \tag{11}$$

where

$$A_{qij} = \delta_{jq}(c_{i,k}v_k + \partial c_i/\partial t) + c_iv_{j,q}. \tag{12}$$

If the curve in question is a vector-line which moves with the fluid, we have $\lambda_q=\phi c_q$, where ϕ is a scalar factor, and also $\partial\Omega_{ij}/\partial t=0$. Hence, by (11),

$$c_q(A_{qij} - A_{qji}) = 0 \tag{13}$$

is a necessary condition for the permanence of vector-lines. It is also sufficient, as we see from consideration of the order of the equations involved and the fact that if (13) holds, then $\Omega_{ij}=0$ implies $\partial\Omega_{ij}/\partial t = 0$.

Hence we have this result: *A necessary and sufficient condition for the permanence of the vector-lines of c_i is* (13), *or equivalently, that the tensor $c_q A_{qij}$, that is,*

$$c_j(c_{i,k}v_k + \partial c_i/\partial t) + c_i v_{j,k} c_k,$$

shall be symmetric.

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NOTE ON A THEOREM IN SUMMABILITY

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Let T denote a regular matrix method of summability in the complex domain, that is to say, a transformation of the form

$$t_n = \sum_{k=1}^{\infty} a_{nk} s_k \qquad (n = 1, 2, 3, \cdots),$$

having the property that the convergence of $\{s_k\}$ to s always implies the existence of t_n for each n and the convergence of $\{t_n\}$ to s. It is well known that the following conditions of Silverman-Toeplitz are necessary and sufficient in order that T be regular: $a_{nk} = o(1)$ $(n \to \infty;$ $k = 1, 2, 3, \cdots)$; $\sum_{k=1}^{\infty} a_{nk} = 1 + o(1)$ $(n \to \infty)$; and

$$\sum_{k=1}^{\infty} |a_{nk}| = O(1) \qquad (n \to \infty). \tag{1}$$

The following theorem was established recently by Henstock [2].[1]

Theorem (Henstock). *Let $y \equiv \{z_k\}$ be a given bounded sequence of complex numbers. Then there exist denumerably many sequences of*

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[1] Numbers in brackets refer to the references at the end of the paper.