CONVEXLY GENERATED k -DIMENSIONAL MEASURES

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1. **Introduction.** Let k and n be integers such that $0 < k \leq n$, and let C_n^k and H_n^k be respectively the k -dimensional measures of Carathéodory and Hausdorff over Euclidean n -space E_n . It is already known that $C_n^k(A) \leq H_n^k(A)$ whenever $A \subseteq E_n$, and that equality holds whenever $k=1$ [1, p. 425],¹ or $k=n$ (by a slight modification of the proof of Sard [4]). The present paper² gives the construction (4.4, 4.7) of a set $A' \subseteq E_3$ such that

$$(1) \quad 0 < C_3^2(A') < H_3^2(A') < \infty.$$

It is further proved that

$$C_3^2(A) \leq H_3^2(A) \leq \frac{\pi(1 + 3^{1/2})}{2} C_3^2(A) \quad \text{whenever } A \subseteq E_3.$$

This result suggests the definition of an equivalence relation (2.1) between measures. This relation had been previously mentioned, without further comment, by Hausdorff [3, p. 158]. It is shown (5.6) that this equivalence relation holds between C_n^k , H_n^k , and certain other k -dimensional measures which are derived by a certain prescribed limiting process (3.2) from functions defined only on certain convex subsets of E_n . This result enables certain proofs by Federer [2] concerning the structure of sets of finite H_n^k measure to be extended to these other measures.

2. Definitions and preliminaries.

2.1. **DEFINITION.** ϕ_1 is congruous to ϕ_2 if and only if ϕ_1 and ϕ_2 are measures over the same set E , and there exist finite positive numbers K_1 , K_2 such that

$$K_1\phi_1(A) \leq \phi_2(A) \leq K_2\phi_1(A) \quad \text{whenever } A \subseteq E.$$

2.2. **REMARK.** Congruity is an equivalence relation.

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¹ Numbers in brackets refer to the references at the end of the paper.

² Another set, constructed by G. Freilich (in publication) at about the same time, was proved by quite different methods to satisfy inequalities similar to (1). No examples were previously known showing $C_n^k \neq H_n^k$, and in fact Federer has shown [2, p. 148] the measures agree on k -rectifiable sets.

2.3. REMARK. If ϕ_1 is congruous to ϕ_2 , then sets of measure zero coincide under the two measures.

3. **Gauges and generated measures.** The measures considered in this paper can all be obtained by applying the same limiting process to various non-negative functions which are defined only on certain subsets of E_n . This general procedure for obtaining a measure is due to Carathéodory [1]. This is formalized below, since the results of this paper are obtained by constructions utilizing properties of the gauges.

3.1. DEFINITION. Given a metric space E , f is a gauge over E if and only if f is such a function that

$$\text{domain } f \subseteq E \text{ [} B \subseteq E \text{]} \quad \text{and range } f \subseteq E_t \text{ [} 0 \leq t \leq \infty \text{]}.$$

3.2. DEFINITION. Given a gauge f over a metric space E , F is generated by f if and only if F is the function defined on

$$E_A \text{ [} A \subseteq E \text{]}$$

such that $F(A)$ is the limit as r approaches zero of the infimum of sums of the form

$$\sum_{j=1}^{\infty} f(B_j)$$

where $B_1, B_2, \dots$ are such members of domain f that $A \subseteq \bigcup_{j=1}^{\infty} B_j$ and diameter $B_j \leq r$ for $j = 1, 2, 3, \dots$.

3.3. THEOREM. If f is a gauge over a metric space, E and F is generated by f , then F is a measure over E and closed sets are F measurable.

3.4. THEOREM. If f is a gauge over a metric space E , domain f is some subset of the set of Borel sets of E , and F is generated by f , then every $A \subseteq E$ is contained in a Borel set of equal F measure.

3.5. DEFINITION. h_n^k is the gauge defined on the set of all compact convex sets $B \subseteq E_n$ by the formula

$$h_n^k(B) = b(k) [\text{diameter } (B)]^k$$

where $b(k)$ is the Lebesgue measure of a sphere of diameter 1 in E_k .

3.6. DEFINITION. c_n^k is the gauge defined on the set of all compact convex sets $B \subseteq E_n$ by the formula

$$c_n^k(B) = \sup_{\pi} (L^k(\pi(B)))$$

where L^k is k -dimensional Lebesgue measure over k -dimensional planes of E_n , and the supremum is taken over the set of all perpendicular projections π of E_n onto k -dimensional planes in E_n .

3.7. DEFINITIONS. Hausdorff measure: H_n^k is generated by h_n^k . Carathéodory measure: C_n^k is generated by c_n^k .

4. 2-dimensional measures over E_3 .

4.1. LEMMA. For any compact convex set $B \subset E_3$ and any $\epsilon > 0$, there exist compact convex sets $B_1, \dots, B_r$ such that

$$(2) \quad B \subseteq \bigcup_{i=1}^r B_i \quad \text{and} \quad \sum_{i=1}^r h_3^2(B_i) \leq \frac{\pi(1+3^{1/2})}{2} c_3^2(B) + \epsilon.$$

PROOF. Let $D = \text{diameter}(B)$. If $D = 0$, take $r = 1$, $B_1 = B$; thus $h_3^2(B_1) = c_3^2(B) = 0$, hence (2) holds. Thus we may assume $D > 0$. Choose $a, b \in B$ such that $|a - b| = D$. Let P_1 be a plane perpendicular to the line joining a and b . Let f be the perpendicular projection function of E_3 onto P_1 . Let $C = f(B)$.

If $\text{diameter}(C) = 0$, then B consists of the line segment of length D joining a to b . Then choose an integer $r > \pi \epsilon D^2 / 4$, and choose r closed subsegments B_i of length D/r covering B , giving

$$\sum_{i=1}^r h_3^2(B_i) = \frac{r\pi(D/r)^2}{4} = \frac{\pi D^2}{4r} < \epsilon,$$

hence (2) holds.

If $0 < K = \text{diameter}(C)$, then choose points $x, y \in C$ such that $|x - y| = K$. Let P_2 be the plane which is projected by f onto the line passing through x and y . Choose $u, v \in B$ such that $f(u) = x$, $f(v) = y$. Then let Q be the set obtained by projecting the union of the triangles (a, u, b) and (a, v, b) perpendicularly into P_2 . Then $L^2(Q) = DK/2$, but since the triangles are subsets of B , we have $c_3^2(B) \geq DK/2$.

Case I. If $0 < (1 + 3^{1/2})K < D$, then let r be the positive integer such that $(r-1)K \leq D < rK$. Since B is contained in a cylinder of cross-section C and altitude D , choose compact convex subcylinders $B_1, B_2, \dots, B_r$, each of altitude K , which cover B . Then for each j , $h_3^2(B_j) \leq (\pi(2^{1/2}K)^2)/4 = \pi K^2/2$. Hence

$$\sum_{i=1}^r h_3^2(B_i) \leq \frac{r\pi K^2}{2}$$

but since $rK \leq D + K < (1 + 1/(1 + 3^{1/2}))D = ((1 + 3^{1/2})D)/2$, we have

$$\sum_{j=1}^r h_3^2(B_j) \leq \frac{\pi K(1+3^{1/2})D}{4} < \frac{\pi(1+3^{1/2})}{2} c_3^2(B).$$

Case II. If $(1+3^{1/2})K \geq D > 0$, then let $r=1$, $B_1=B$, and $h_3^2(B_1) = (\pi D^2)/4 \leq (\pi DK(1+3^{1/2}))/4 \leq (1+3^{1/2})(\pi/2)c_3^2(B)$, hence (2) holds.

4.2. THEOREM. If $A \subseteq E_3$, then

$$C_3^2(A) \leq H_3^2(A) \leq \frac{\pi(1+3^{1/2})}{2} C_3^2(A).$$

PROOF. The left-hand inequality is a previously known immediate consequence of the isodiametric inequality, and the right-hand inequality follows immediately from 3.2, 3.5, 3.6, 3.7, and 4.1.

4.3. THEOREM. If $0 \leq k \leq n$, $0 \leq i \leq m$, $A \subseteq E_n$, $0 < M$, and f is a function mapping A into E_m satisfying the condition $|f(x) - f(y)| \leq M|x - y|^{k/i}$ whenever $x, y \in A$, then $H_n^k(A) \geq (b(k)/M^i b(i)) H_m^i(\text{range } f)$.

PROOF. Given any sequence $B_1, B_2, \dots$ as in 3.2, then the sequence $C_1, C_2, \dots$ such that C_j is the closure of the convex hull of $f(B_j)$ satisfies $\text{range } f \subseteq \bigcup_{j=1}^\infty C_j$, diameter $C_j < Mr^{k/i}$ for all j , and

$$\begin{aligned} h_m^i(C_j) &\leq b(i) [\text{diameter } (C_j)]^i \leq b(i) (M[(\text{diameter } (B_j))^{k/i}])^i \\ &\leq \frac{M^i b(i)}{b(k)} h_n^k(B_j), \end{aligned}$$

hence the conclusion follows by summation over j and passage to the limit.

4.4. DEFINITION. Let A' be the set of all those points of E_3 whose three coordinates are between 0 and 1 and have an octal expansion in which only the digits 0, 1, 2, 3 occur.

4.5. THEOREM. $0 < H_3^2(A') < +\infty$.

PROOF. Let G be the set of all numbers of the form

$$(3) \quad \sum_{i=1}^{\infty} a_i \cdot 8^{-i},$$

$a_i = 0, 1, 2$, or 3 for $i = 1, 2, 3, 4, \dots$. Since the digit 7 is excluded, every point of G has a unique representation in the form (3).

Let f be the function on G onto the closed unit interval such that

$$f\left(\sum_{i=1}^{\infty} a_i \cdot 8^{-i}\right) = \sum_{i=1}^{\infty} a_i \cdot 4^{-i}.$$

Then if $u, v \in G$ and $u > v$, let

$$u = \sum_{i=1}^{\infty} a_i \cdot 8^{-i}, \quad v = \sum_{i=1}^{\infty} b_i \cdot 8^{-i}, \quad m = \inf_i E[a_i > b_i];$$

then

$$\begin{aligned} u - v &= \sum_{i=1}^{\infty} (a_i - b_i) \cdot 8^{-i} \geq (a_m - b_m) \cdot 8^{-m} - \sum_{i=m+1}^{\infty} 3 \cdot 8^{-i} \\ &\geq 8^{-m} - (3/7) \cdot 8^{-m} = (4/7) \cdot 8^{-m}, \\ f(u) - f(v) &= \sum_{i=1}^{\infty} (a_i - b_i) \cdot 4^{-i} \leq (a_m - b_m) \cdot 4^{-m} + \sum_{i=m+1}^{\infty} 3 \cdot 4^{-i} \\ &\leq 3 \cdot 4^{-m} + 4^{-m} = 4 \cdot 4^{-m}. \end{aligned}$$

Thus

$$|f(u) - f(v)| \leq (14)^{2/3} |u - v|^{2/3} \quad \text{for all } u, v \in G.$$

Let g be the function on A' onto the unit cube J in E_3 such that

$$g(x) = (f(x_1), f(x_2), f(x_3)) \quad \text{for all } x = (x_1, x_2, x_3) \in A'.$$

Then g satisfies the Lipschitz condition

$$|g(x) - g(y)| \leq 14^{2/3} |x - y|^{2/3} \quad \text{for all } x, y \in A',$$

and since, by [4], $H_3^2(J) = 1$, we conclude from 4.3 that $H_3^2(A') > 0$.

To obtain the other inequality, for each $r > 0$, choose m so that $8^{-m} 3^{1/2} < r$ and note that A' is contained in the 64^m cubes B_i with side length 8^{-m} , corresponding to the $64^m = 4^{3m}$ admissible combinations of the first m digits of the coordinates. Each of these cubes has diameter $8^{-m} 3^{1/2} < r$, and

$$\sum_{i=1}^{64^m} h_3^2(B_i) \leq \frac{\pi(3^{1/2} \cdot 8^{-m})^2}{4} \cdot 64^m = 3\pi/4,$$

hence $H_3^2(A') \leq 3\pi/4 < \infty$.

4.6. LEMMA. If A^* is a closed subset of A' , and A is the convex hull of A^* , then

$$c_3(A) \leq (1 - 2 \cdot 10^{-10}) h_3^2(A).$$

PROOF. Assume A has positive diameter D . Let P be any plane and let π be the perpendicular projection function of E_3 onto P . Let $\pi(A) = B$.

We wish to show that

$$(4) \quad L^2(B) \leq \frac{\pi D^2}{4} (1 - 2 \cdot 10^{-10}).$$

Let Z be the boundary of B in P . Then Z is connected. Since $A \cap \text{inverse } \pi(x)$ is a line segment for each $x \in Z$, the set $S = A \cap \text{inverse } \pi(Z)$ is a continuum. Choose $a, b \in S$ such that $|a - b| > 3^{1/2}D/2$. (If no such points exist, (4) is obviously satisfied.) Then choose i so that $|a_i - b_i| \geq |a_j - b_j|$ for $j = 1, 2, 3$, where a_j is the j th coordinate of a . Then $|a - b| \leq 3^{1/2}|a_i - b_i|$, hence

$$|a_i - b_i| > \frac{1}{2} \left(3^{1/2}D \frac{1}{3^{1/2}} \right) = \frac{1}{2}D.$$

For the above choice of i , let m be the least integer such that there exist points $x, y \in S$ for which $3 \cdot 8^{-m}/2 \leq x_i - y_i$. Then $|a_i - b_i| < 3 \cdot 8^{-(m-1)}/2$, hence $D/24 < 8^{-m}$. Fix x, y satisfying the above condition. If the $(m+1)$ th digit in the octal expansion of y_i is 0, 1, 2, or 3, let K be the set of all points z such that the first m digits of the octal expansion of z_i agree with those of y_i and the $(m+1)$ th digit of the octal expansion of z_i is 4, 5, 6, or 7. If the $(m+1)$ th digit of the octal expansion of y is 4, 5, 6, or 7, let K be the set of all points z such that the first m digits of the octal expansion of z_i agree with those of $(y_i + 8^{-m})$ and the $(m+1)$ th digit of z_i is 4, 5, 6, or 7. In either case, K is the strip between two parallel planes whose distance apart is $8^{-m}/2$, $A^* \cap (\text{interior } K)$ is empty, and x and y are on opposite sides of K .

Now consider the plane P_1 which is parallel to boundary K and bisects K . Since S is connected we can choose a point $z \in (S \cap P_1)$. Then distance $(z, A^*) \geq 8^{-m}/4 > D/96$.

Since $\pi(z) \in Z$, choose a line J in P such that J is a supporting line of B at $\pi(z)$. Then let $P_2 = \text{inverse } \pi(J)$. Then P_2 is a supporting plane of A at z . But since A is the convex hull of A^* , and distance $(z, A^*) > D/96$, there exists a point $w \in (P_2 \cap A^*)$ such that $|z - w| > D/96$. Hence if r is the midpoint of the line segment from z to w , then $|r - z| = |r - w| > D/192$. By the definition of D we have $|r - s| \leq D$ and $|w - s| \leq D$ for all $s \in A$. Supposing that X_1 and X_2 measure the angles at r of the triangles (r, z, s) and (r, w, s) respectively, we have, by the law of cosines,

$$D^2 \geq |z - s|^2 = |r - s|^2 + D^2/192^2 - (2D/192)|r - s|\cos(X_1),$$

$$D^2 \geq |w - s|^2 = |r - s|^2 + D^2/192^2 - (2D/192)|r - s|\cos(X_2),$$

but since X_1 and X_2 are supplementary, we have $\cos(X_2) = -\cos(X_1)$, hence

$$D^2 \geq |r - s|^2 + D^2/36864,$$

$$|r - s| \leq D \left(\frac{36863}{36864} \right)^{1/2} < D \frac{99,999}{100,000}.$$

Since this is true for any $s \in A$, we conclude by projecting into the plane P_1 that $\sup_{x \in B} |q - x| < D(99,999/100,000)$, where $q = \pi(r)$, and since $r \in P_2$ we have $q \in J$. Let Q be a half-disc of radius $D/100,000$ and center q bounded by J and lying outside B . Then $\sup_{m \in Q, x \in B} |m - x| < D(99,999/100,000) + D/100,000 = D$. But since we already have diameter $B \leq D$, this gives diameter $(B \cup Q) \leq D$. By the isodiametric inequality, we have $L^2(B \cup Q) \leq \pi D^2/4$. Hence $L^2(B) + L^2(Q) \leq \pi D^2/4$, and since $L^2(Q) = \pi D^2 \cdot 10^{-10}/2$, we conclude $L^2(B) \leq (\pi D^2/4)(1 - 2 \cdot 10^{-10})$.

4.7. THEOREM. $0 < C_3^2(A') \leq (1 - 2 \cdot 10^{-10}) H_3^2(A') < \infty$.

PROOF. This follows from 3.2, 3.5, 3.6, 3.7, 4.2, 4.5, and 4.6.

5. **The general case.** Let $k \leq n$ be positive integers. For each bounded convex set $B \subset E_n$, let $M(B)$ be the supremum of $L^n(S)$, where S is an n -simplex contained in closure B , and let $m(B)$ be the infimum of $L^n(I)$, where I is a closed rectangular parallelepiped (whose faces are not necessarily parallel to the coordinate planes) containing B . The fact that these extremal values are attained simplifies the proofs which follow.

5.1. LEMMA. If B is a bounded convex subset of E_n , then

$$m(B) \leq n! 2^{n-1} M(B).$$

PROOF. By induction on n . If $n = 1$, then $m(B) = M(B) = L^1(B)$. Given that the conclusion is true for E_{n-1} , consider any bounded convex set $B \subset E_n$. Let S' be an $(n-1)$ -simplex contained in closure B which is such that $L^{n-1}(S')$ is as large as possible. Let P be the $(n-1)$ -plane which contains S' . Let π be the perpendicular projection of E_n onto P . Let $B' = \pi(\text{closure } B)$. By identifying P with E_{n-1} , we can apply the inductive hypothesis to B' , obtaining a rectangular parallelepiped I' and an $(n-1)$ -simplex S'' such that $S'' \subseteq B' \subseteq I' \subseteq P$ and $L^{n-1}(I') \leq (n-1)! 2^{n-2} L^{n-1}(S'')$.

Let S^* be an $(n-1)$ -simplex spanned by some set X such that $X \subset \text{closure } B$, cardinal $(X) \leq n$, and $\pi(X)$ is the set of vertices of S'' . Then $L^{n-1}(S') \geq L^{n-1}(S^*) \geq L^{n-1}(S'')$, hence $S' \subseteq \text{closure } B$ and S' satisfies $L^{n-1}(I') \leq (n-1)! 2^{n-2} L^{n-1}(S')$.

Let $q = \sup_{x \in B} (\text{distance } (x, P))$. Choose $w \in \text{closure } B$ such that distance $(w, P) = q$. Let S be the n -simplex spanned by $S' \cup \{w\}$,

giving $S \subseteq \text{closure } B$ and $L^n(S) = (q/n)L^{n-1}(S')$, hence $L^{n-1}(I') \leq n!2^{n-2}L^n(S)/q$.

Let $I = (E_x [\text{distance } (x, P) \leq q] \cap \text{inverse } \pi(I'))$. Then $I \supseteq B$, I is a closed rectangular parallelepiped, and $L^n(I) = 2qL^{n-1}(I')$, hence $L^n(I) \leq n!2^{n-1}L^n(S)$. The conclusion then follows from the definitions of $m(B)$ and $M(B)$.

5.2. LEMMA. Suppose

1. $k \leq n$ are positive integers,
2. f is a gauge whose domain is some subset of the set of all bounded convex subsets of E_n , and there exists a finite positive number K such that $f(B) \geq KL^k(S)$ whenever S is a k -simplex of E_n , $S \subseteq \text{closure } B$, and $B \in \text{domain } f$,
3. g is such a gauge that there exists a finite positive number N such that given any rectangular parallelepiped $I \subseteq E_n$ such that s is the length of the longest side of I , then there exists a set $C \supseteq I$ such that $\text{diameter } C \leq N \cdot s$ and $g(B) \leq N \cdot s^k$,
4. $B \in \text{domain } f$, $\epsilon > 0$.

Then there exist $B_1, \dots, B_r$ such that

$$B \subseteq \bigcup_{i=1}^r B_i, \quad \sum_{i=1}^r g(B_i) \leq \frac{2N}{k!K} f(B) + \epsilon,$$

and $\text{diameter } B_j \leq Nn^{1/2} \cdot \text{diameter } B$ for $j=1, \dots, r$.

PROOF. Choose a rectangular parallelepiped I such that $I \supseteq \text{closure } B$ and $L^n(I) = m(\text{closure } B)$. For $i=1, \dots, n$ let a_i be the length of the edges of I which extend in the i th direction. Choose the order of enumerating these directions in such a way that $a_1 \geq a_2 \geq \dots \geq a_{k-1} \geq a_k \geq \dots \geq a_n \geq 0$.

Case I. If $a_k = 0$, then $a_j = 0$ for $j \geq k$. Hence I is a degenerate parallelepiped of dimension less than or equal to $k-1$. Choose an integer $s > a_1^k / N\epsilon$. Then subdivide I into $r = s^{k-1}$ rectangular parallelepipeds I_j such that each of their edges have length $1/s$ times the length of the corresponding edge of I , hence for each j , there exists a set $B_j \supseteq I_j$ such that $g(B_j) \leq N \cdot (a_1/s)^k$, hence

$$\sum_{j=1}^{s^{k-1}} g(B_j) \leq \frac{N}{s} a_1^k < \epsilon.$$

Case II. If $a_k > 0$, then $a_i \geq a_k > 0$ for $1 \leq i \leq k$. Let π be the perpendicular projection function which maps E_n onto a k -plane P which contains a face I' of I such that the edges of I' have lengths $a_1, \dots, a_k$. Let $B' = \pi(\text{closure } B)$. Then I' is a rectangular parallele-

pipied of P containing B' such that $L^k(I')$ is minimal, for otherwise this would contradict the way in which I was chosen. Then applying 5.1 we obtain a k -simplex $S' \subseteq B'$ such that $L^k(S') \geq L^k(I')/(k!2^{k-1})$.

Choose a set $X \subseteq \text{closure } B$ such that $\text{cardinal}(X) \leq n$ and $\pi(X)$ is the set of vertices of S' . Then the k -simplex S spanned by X has the property that

$$L^k(S) \geq L^k(S') \geq L^k(I')/(k!2^{k-1}) \geq a_1 \cdot a_2 \cdot \dots \cdot a_k/(k!2^{k-1}).$$

Then since $S \subseteq \text{closure } B$, we have $f(B) \geq K \cdot a_1 \cdot \dots \cdot a_k/(k!2^{k-1})$.

For $i=1, \dots, k$, let r_i be the positive integer such that $(r_i-1)a_k < a_i \leq r_i a_k$. For each such i , $r_i a_k < 2a_i$. Subdivide the parallelepiped I' into $r = r_1 \cdot r_2 \cdot \dots \cdot r_k$ rectangular parallelepipeds $I_1, \dots, I_r$ such that the i th edge of each parallelepiped I_j has length $a_i/r_i \leq a_k$. For each j , let $A_j = I_j \cap (\text{inverse } \pi(I_j))$. Then for each j , A_j is a rectangular parallelepiped whose longest edge has length a_k . Hence there exist sets $B_j \supseteq A_j$ such that $g(B_j) \leq N a_k^k$. Then

$$\begin{aligned} \sum_{j=1}^r g(B_j) &\leq r \cdot N a_k^k \leq N \prod_{i=1}^k r_i a_k \leq N \cdot 2^k \prod_{i=1}^k a_i \\ &\leq \frac{N 2^k f(B)}{K(k!2^{k-1})} = \frac{2N}{k!K} f(B). \end{aligned}$$

5.3. LEMMA. If 1, 2, and 3 of 5.2 hold, F is generated by f , and G is generated by g , then

$$F(A) \geq \frac{k!K}{2N} G(A) \quad \text{whenever } A \subseteq E_n.$$

PROOF. This follows from 3.2 and 5.2.

5.4. DEFINITION. F is a convexly generated k -dimensional measure over E_n if and only if there exists a gauge f such that

- (1) F is generated by f ;
- (2) Domain f is some subset of the set of all convex Borel subsets of E_n ;
- (3) There exists a finite positive number K such that $f(B) \geq K L^k(S)$ whenever S is a k -simplex of E_n , $S \subseteq \text{closure } B$, and $B \in \text{domain } f$;
- (4) There exists a finite positive number N such that whenever I is a rectangular parallelepiped in E_n such that s is the length of the longest edge of I , then there exists a set $C \supseteq I$ such that $\text{diameter } C \leq Ns$ and $f(C) \leq Ns^k$.

5.5. REMARK. H_n^k and C_n^k are convexly generated k -dimensional measures over E_n . Also sphere measure $\mathbb{S}_n^k$ and Gillespie measure $\mathcal{G}_n^k$

as defined by Federer [2, p. 122], together with several of the measures derived from other gauges suggested by Hausdorff [3], are convexly generated k -dimensional measures.

5.6. THEOREM. *If ϕ_1 and ϕ_2 are convexly generated k -dimensional measures over E_n , then ϕ_1 is congruous to ϕ_2 , and A is ϕ_1 measurable if and only if A is ϕ_2 measurable.*

PROOF. Apply 5.3 twice to obtain the congruity. The equivalence of measurability follows from 2.3, 3.3, and 3.4.

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