SOME LIMITS OF BOOLEAN ALGEBRAS

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1. Introduction. Inverse systems of topological spaces and direct and inverse systems of Abelian groups and resulting applications thereof to algebraic topology have been studied intensively, and the techniques employed are now standard [2]. (Numbers in brackets refer to the bibliography at the end of the paper.) We shall apply some of these ideas to the treatment of sets of Boolean algebras in order to obtain, in the inverse case, a representation, and, in both cases, some information about the ideals and about the decompositions of certain algebras. For instance, it is well known [1] that the cardinal product B^* of a set $\{B_\alpha\}$ of Boolean algebras, indexed by a class A , is also a Boolean algebra. Moreover, if each B_α is m -complete for some fixed cardinal m [3], then B^* is m -complete. When A is a set partly ordered by $\leq$ and directed thereby, and when the B_α form an inverse system (to be defined below), we shall show that the resulting inverse limit set is a Boolean algebra, a subalgebra of B^* , and a similar result will hold in the direct case, at least in part.¹ For each nonfinite Boolean algebra B , there will be constructed a nontrivial inverse limit representation of B , the inverse limit of all principal ideals [3] of B (with repetitions). In the direct case, ideals in the limit algebra (and the homomorphic algebras which they generate) will turn out to be direct limits of like objects. Direct decomposition into ideals of the direct limit takes place (it will be seen) if there is a similar decomposition in each B_α .

Following a suggestion of the referee, we shall investigate the underlying Stone representation spaces of the algebras involved. It turns out that spaces associated with an inverse limit of algebras have a direct limit which is a Stone subspace of the Stone space of the inverse limit. On the other hand, the spaces associated with a direct limit of algebras have an inverse limit Stone space which is homeomorphic to the Stone space of the direct limit.

A Boolean algebra B is defined to be a complemented distributive lattice [1] with partial ordering relation $\leq$. Complements are denoted by primes, and the cup ($\cup$)-cap ($\cap$) notation of [1] is used practically without modification.

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¹ See S. Bochner, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 439-443

2. The inverse limit. Let A be a partly ordered set with order relation $\leq$ which directs A in that $\alpha, \beta \in A$ imply the existence of $\gamma \in A$ with $\alpha, \beta \leq \gamma$. Let $\{B_\alpha\}$ be a set of Boolean algebras indexed by A . For each inequality $\beta \leq \alpha$ ($\alpha, \beta \in A$), suppose that there exists a lattice homomorphism $[1] \Pi_{\beta\alpha}$ of B_α into B_β in such a way that (1) if $I_\beta, 0_\beta$ ($I_\alpha, 0_\alpha$) are the supremum and infimum of B_β (B_α), then $\Pi_{\beta\alpha} I_\alpha = I_\beta$, and $\Pi_{\beta\alpha} 0_\alpha = 0_\beta$; and (2) $\gamma \leq \beta \leq \alpha$ ($\gamma, \beta, \alpha \in A$) imply $\Pi_{\gamma\alpha} = \Pi_{\gamma\beta} \Pi_{\beta\alpha}$. We also require that $\Pi_{\alpha\alpha}$ be the identity mapping of B_α onto B_α . Such a set of Boolean algebras and corresponding mappings will be called an inverse system (of Boolean algebras). Let B^* be the cardinal product of the B_α . Let $x \in B^*$ have component $x_\alpha \in B_\alpha$. For $x, y \in B^*$, $x \leq y$ means that $x_\alpha \leq y_\alpha$ for every $\alpha \in A$. Let a subset B^i of B^* be defined as follows: $x \in B^i$ if $x \in B^*$ and if, for each inequality $\beta \leq \alpha$, $\Pi_{\beta\alpha} x_\alpha = x_\beta$. B^i has a hereditary partial order from B^* . The element $I = \{I_\alpha\} \in B^i$ is the supremum of B^i , and $0 = \{0_\alpha\}$ is the infimum. It is trivial to show that under the ordering hereditary from B^* , B^i is a Boolean algebra, a subalgebra of B^* . We might call B^i the inverse limit of the factors B_α and write $B^i = \lim_{\leftarrow} B_\alpha$.

Let m be a fixed infinite cardinal. Let $\{B_\alpha\}$ be an inverse system of Boolean algebras with mappings $\Pi_{\beta\alpha}$ and inverse limit B^i . Let each $\Pi_{\beta\alpha}$ have the property that if $x_\alpha \in B_\alpha$ is the least upper bound (dually, greatest lower bound) of the elements of a subset S of B_α , where power $S \leq m$, then $\Pi_{\beta\alpha} x_\alpha = \text{l.u.b. (g.l.b.) of the elements of } \Pi_{\beta\alpha} S$. Let such a set be called an m -potent inverse system.

THEOREM 1. *Let $\{B_\alpha\}$ be an m -potent inverse system of Boolean algebras with inverse limit B^i . Let Γ be a cofinal subset of A such that for $\gamma \in \Gamma$, B_γ is m -complete. Then B^i is an m -complete Boolean algebra.*

PROOF. Let $\{x^\sigma\}$ be a class $S \subset B^i$, where $\sigma \in \Sigma$, an index class of power less than or equal to m . For given $\alpha \in A$, there exists $\gamma \in \Gamma$ such that $\alpha \leq \gamma$, and $\Pi_{\alpha\gamma} x_\gamma^\sigma = x_\alpha^\sigma$. Let $x_\gamma \in B_\gamma$ be $\bigvee_\sigma x_\gamma^\sigma$ (the l.u.b. of the x_γ^σ). Then $x_\alpha = \Pi_{\alpha\gamma} x_\gamma = \bigvee_\sigma x_\alpha^\sigma$, by the hypothesis on the $\Pi_{\alpha\gamma}$. Define $x \in B^*$ by $x = \{x_\alpha\}$. For each inequality $\beta \leq \alpha$, there exists $\delta \in \Gamma$ with $\beta \leq \alpha \leq \delta$, and $\Pi_{\alpha\delta} x_\delta = x_\alpha$, $\Pi_{\beta\delta} x_\delta = x_\beta$. Hence $\Pi_{\beta\alpha} x_\alpha = \Pi_{\beta\alpha} \Pi_{\alpha\delta} x_\delta = \Pi_{\beta\delta} x_\delta = x_\beta$, and $x = \{x_\alpha\} = \bigvee_\sigma x^\sigma \in B^i$, and dually for $\bigwedge_\sigma x^\sigma$.

3. The representation theorem. Let B be a Boolean algebra. For $y \in B$, the principal ideal (y) is the set of all $x \in B$ such that $x \leq y$. For $x \leq y$, $x, y \in B$, let ϕ_{xy} be a mapping with domain (y) into $(x) \subset (y)$ such that for $z \in (y)$, $\phi_{xy} z = z \cap x \in (x)$. It is trivial to show that ϕ_{xy} is a homomorphism of (y) onto (x) which preserves all (not only finite) l.u.b.'s and g.l.b.'s. From $x \leq y \leq z$, $x, y, z \in B$, we have $(x) \subset (y) \subset (z)$

and $\phi_{xz} = \phi_{xy}\phi_{yz}$. Thus, the set of Boolean algebras $\{(x)\}$, $x \in B$, is an inverse system. Call its inverse limit C . Clearly, $C \cong B$; for, if (I) is the principal ideal generated by the supremum I of B , then $(I) = B$. But (I) is maximum in the set of sets $\{(x)\}$ under the ordering relation of inclusion. C has a projection onto the component (I) , and it is easily seen that this projection is the required isomorphism.

Let $\mathcal{P}$ be a set consisting of all finite-membered subsets P_α of B , where each P_α is a directed set under the order relation $\leq$ of B . $\mathcal{P}$ is directed by inclusion. For, given P_α and $P_\beta \in \mathcal{P}$, construct $P_\gamma \in \mathcal{P}$ as the set sum of P_α , P_β , and I , the supremum of B . Then $P_\alpha, P_\beta \subset P_\gamma$. Since each P_α is finite, there exists a greatest element $x_\alpha \in P_\alpha$. Let B_α be the inverse limit algebra of the ideals (y_α) which are generated by the elements $y_\alpha \in P_\alpha$. Then $B_\alpha \cong (x_\alpha)$. Suppose that $P_\alpha \subset P_\beta$ with maximum elements x_α and x_β , respectively. Let $\{y'_\beta\}$ be an element of B_β , where $t \in P_\beta$. Construct $\{y'_\alpha\}$ in B_α^* , the cardinal product of the factors of B_α , by putting $y'_\alpha = y'_\beta$ for every $t \in P_\alpha$. $\{y'_\beta\} \in B_\alpha$, since $x_\alpha \leq x_\beta$, where $B_\alpha \cong (x_\alpha)$, $B_\beta \cong (x_\beta)$. Define $\Pi_{\alpha\beta}$, a mapping from B_β into B_α , by $\Pi_{\alpha\beta}\{y'_\beta\} = \{y'_\alpha\}$. It is easy to show that $\Pi_{\alpha\beta}$ is a lattice homomorphism of B_β onto B_α and that if $P_\alpha \subset P_\beta \subset P_\gamma$, then $\Pi_{\alpha\gamma} = \Pi_{\alpha\beta}\Pi_{\beta\gamma}$.

Let D be the inverse limit algebra of the B_α 's. Consider those P_α , and associate B_α , for which $I \in P_\alpha$, where I is the supremum of B . The set S of such P_α is nonempty and cofinal in the set of all P_α as ordered by inclusion. $P_\alpha \in S$ implies $B_\alpha \cong B$. Note that there is no maximum element in S (or in the set $\mathcal{P}$ of all P_α) if B is infinite. Let Π_α be the projection of D into a factor B_α associated with a $P_\alpha \in S$. It is routine to show that Π_α is onto B_α and that Π_α is an isomorphism. Since $B_\alpha \cong B$, then $D \cong B$. We have proved the following theorem.

THEOREM 2. *Let B be an infinite Boolean algebra. B is isomorphic to the inverse limit of an inverse system of Boolean algebras. Each algebra in this system is isomorphic to a principal ideal of B , and there is no final algebra in the system. Other than the algebra (0) , each such ideal occurs with repetitions, and each principal ideal of B appears. Under the ordering relation of the factors of the inverse system each factor is covered by an algebra isomorphic to B .*

4. The direct limit. Let A be a set partly ordered by $\leq$ and directed. Let $\{B_\alpha\}$ be a set of Boolean algebras indexed by A . For each inequality $\beta \leq \alpha$, $\alpha, \beta \in A$, suppose that there exists a lattice homomorphism $\phi_{\beta\alpha}$ of B_β into B_α in such a way that (1) $\phi_{\beta\alpha}I_\beta = I_\alpha$ and $\phi_{\beta\alpha}0_\beta = 0_\alpha$; and (2) $\gamma \leq \beta \leq \alpha$ ($\gamma, \beta, \alpha \in A$) imply $\phi_{\gamma\alpha} = \phi_{\beta\alpha}\phi_{\gamma\beta}$. $\phi_{\alpha\alpha}$ is to be the identity mapping of B_α onto B_α . We can define an equivalence

relation on the ordered pairs (α, x_α) ($\alpha \in A, x_\alpha \in B_\alpha$) by writing $(\alpha, x_\alpha) \sim (\beta, x_\beta)$ if there exists $\gamma \geq \alpha, \beta$ with $\phi_{\alpha\gamma}x_\alpha = \phi_{\beta\gamma}x_\beta$. Let B^d be the set of all equivalence classes of such ordered pairs. Such classes are denoted by $\{(\alpha, x_\alpha)\}$ and can be ordered in a natural way by writing $\{(\alpha, x_\alpha)\} \leq \{(\beta, x_\beta)\}$ if there exist $(\alpha_1, x_{\alpha_1}) \sim (\alpha, x_\alpha)$ and $(\beta_1, x_{\beta_1}) \sim (\beta, x_\beta)$ and a $\gamma \in A, \gamma \geq \alpha_1, \beta_1$, such that $\phi_{\alpha_1\gamma}x_{\alpha_1} \leq \phi_{\beta_1\gamma}x_{\beta_1}$. It can be proved that the relation $\leq$ does not depend on the particular representatives of the equivalence classes which are employed. That is, there exists $\gamma \geq \alpha, \beta$ with $\phi_{\alpha\gamma}x_\alpha \leq \phi_{\beta\gamma}x_\beta$. Under this ordering, B^d is a Boolean algebra, the element $\{(\alpha, I_\alpha)\}$ is the supremum of B^d , and $\{(\alpha, 0_\alpha)\}$ is the infimum. An element $\{(\alpha, x_\alpha)\}$ has the complement $\{(\alpha, x'_\alpha)\}$, where x'_α is the complement of x_α in B_α . This Boolean algebra may be called the direct limit algebra of the B_α and may be denoted by $B^d = \lim_{\rightarrow} B_\alpha$.

5. An example of a direct limit. Let X be a compactum which is, in at least one way, a homogeneous simplicial polyhedron α of dimension n with only a finite number of n -faces. For a given finite, n -homogeneous, simplicial polyhedral decomposition α of X , the set of all simplicial subpolyhedra of dimension n forms a Boolean algebra B_α in an obvious fashion if one admits the null polyhedron as an n -homogeneous subpolyhedron of every such subpolyhedron of X . Write $\alpha \leq \beta$ if β is a refinement of α . That is, every closed n -face of β is included in a closed n -face of α . For $x_\alpha \in B_\alpha$ and $\alpha \leq \beta$, define $\phi_{\alpha\beta}x_\alpha = x_\beta \in B_\beta$ by taking x_β to be that subpolyhedron of β , the point set of which is the set sum of all n -faces of β which lie in some face of x_α . $\phi_{\alpha\beta}$ is a lattice homomorphism of B_α into B_β in such a way that the null set is carried onto the null set, and α is carried onto β , the respective suprema of B_α and B_β . Given α and β , there exists a common refinement γ , a polyhedral covering of X of the given type. It turns out that if $\alpha \leq \beta \leq \gamma$, then $\phi_{\alpha\gamma} = \phi_{\beta\gamma}\phi_{\alpha\beta}$. Therefore, the B_α with mappings $\phi_{\alpha\beta}$ are a direct system of Boolean algebras. The limit algebra B^d is the algebra of all polyhedra which are subsets of X and is an atomless Boolean algebra. (For instance, use barycentric subdivision to obtain a refinement of any given element.)

6. Ideals.

THEOREM 3. Let $B^d = \lim_{\rightarrow} B_\alpha$. (1) For a given ideal J in B^d , the set of all $x_\alpha \in B_\alpha$ for which $\{(\alpha, x_\alpha)\} \in J$ is an ideal J_α in B_α . Let the $\phi_{\alpha\beta}$ be onto mappings. (2) For given $x \in J$ and $\alpha \in A$, there exists $x_\alpha \in J_\alpha$ with $\{(\alpha, x_\alpha)\} = x$ (so that the J_α are nonvoid and are "representative" of J). (3) For $\alpha \leq \beta$, $\phi_{\alpha\beta}J_\alpha = J_\beta$ and $\phi_{\alpha\beta}^{-1}J_\beta = J_\alpha$. (4) $J = \lim_{\rightarrow} J_\alpha$ with suitably restricted mappings $\phi_{\alpha\beta}$.

PROOF. (1), (2), and (3) are obvious. (4) Let M be the set of all $\{(\alpha, x_\alpha)\} \in \lim_{\rightarrow} B_\alpha$ for which there exist representatives (α, x_α) with $x_\alpha \in J_\alpha$. If $(\beta, x_\beta) \sim (\alpha, x_\alpha)$, there exists $\gamma \geq \alpha, \beta$ with $\phi_{\alpha\gamma}x_\alpha = \phi_{\beta\gamma}x_\beta$, and $\phi_{\beta\gamma}x_\beta \in J_\gamma$, $x_\beta \in J_\beta$, by (3). It is clear that $M = J$, but $M = \lim_{\rightarrow} J_\alpha$ with the mappings $\phi_{\alpha\beta}$ restricted to the J_α .

THEOREM 4. Let $B^d = \lim_{\rightarrow} B_\alpha$, and let the $\phi_{\alpha\beta}$ be onto mappings. For given $m \in B^d$, let M_α be the set of all $m_\alpha \in B_\alpha$ such that $\{(\alpha, m_\alpha)\} = m$. (1) For $\alpha \leq \beta$, $\phi_{\alpha\beta}M_\alpha = M_\beta$ and $\phi_{\alpha\beta}^{-1}M_\beta = M_\alpha$. (2) Suppose that there exists a selection function f on A into the M_α such that $f(\alpha) = m_\alpha \in M_\alpha$ in such a way that $\alpha \leq \beta$ implies $\phi_{\alpha\beta}m_\alpha = m_\beta$. Then for the principal ideals (m_α) of B_α with suitably restricted mappings $\phi_{\alpha\beta}$, $\lim_{\rightarrow} (m_\alpha)$ exists and equals (m) , the principal ideal in B^d with supremum m . (3) Let the B_α be complete. Then the completions $(m_\alpha)^-$ of the (m_α) are principal ideals in B_α , and $(m)^- = (\bigvee m_\alpha)$, where the l.u.b. is taken over all $m_\alpha \in M_\alpha$. (4) Let the B_α be complete, and let the $\phi_{\alpha\beta}$ preserve all least upper bounds. Then $\phi_{\alpha\beta}(m)^- = (m)^-$.

PROOF. (1) is obvious. (2) $\phi_{\alpha\beta}$ restricted to (m_α) is into (m_β) . If $x_\beta \leq m_\beta$, there exists $x_\alpha^* \in B_\alpha$ such that $\phi_{\alpha\beta}x_\alpha^* = x_\beta$. Let $x_\alpha = x_\alpha^* \cap m_\alpha$. Then $\phi_{\alpha\beta}x_\alpha = x_\beta \cap m_\beta = x_\beta$, so that $\phi_{\alpha\beta}$ is onto (m_β) . From this, $N = \lim_{\rightarrow} (m_\alpha)$ exists and is a subset of (m) . For $\{(\alpha, y_\alpha)\} \in (m)$, $n_\alpha = y_\alpha \cup m_\alpha \in M_\alpha$. There exists $\gamma \in A$, $\gamma \geq \alpha$, such that $\phi_{\alpha\gamma}n_\alpha = \phi_{\alpha\gamma}m_\alpha$. Then $\{(\alpha, y_\alpha)\} = \{(\gamma, \phi_{\alpha\gamma}y_\alpha)\}$. But $\phi_{\alpha\gamma}y_\alpha \leq \phi_{\alpha\gamma}m_\alpha = m_\gamma$. Hence $\{(\alpha, y_\alpha)\} \in N$.

(3) $\bigvee m_\alpha$ exists, by the completeness of B_α , and lies in $(m)^-$. If $\{(\alpha, x_\alpha)\} \in (m)$, there exists $n_\alpha \in M_\alpha$ with $x_\alpha \leq n_\alpha$, as we see from the proof of (2). Hence $\bigvee x_\alpha \leq \bigvee m_\alpha$, where the first l.u.b. is taken over all $x_\alpha \in (m)_\alpha$, and the second is taken over all $m_\alpha \in M_\alpha$. Hence $\bigvee x_\alpha = \bigvee m_\alpha$, and the latter is the generator of the principal ideal $(m)^-$ in B_α .

(4) If $x_\alpha \in (m)_\alpha^-$, then $x_\alpha \leq \bigvee m_\alpha$, $\phi_{\alpha\beta}x_\alpha \leq \bigvee \phi_{\alpha\beta}m_\alpha = \bigvee m_\beta$, so that $\alpha \leq \beta$ implies $x_\beta = \phi_{\alpha\beta}x_\alpha \in (m)_\beta^-$. For $x_\beta \in (m)_\beta^-$, $x_\beta \leq \bigvee_\mu m_\beta^\mu$, $m_\beta^\mu \in M_\beta$. There exist $x_\beta^\mu \leq m_\beta^\mu$ such that $x_\beta = \bigvee_\mu x_\beta^\mu$, and there exist x_α^μ such that $\phi_{\alpha\beta}x_\alpha^\mu = x_\beta^\mu$. Hence $x_\alpha^\mu \in (m)_\alpha$, and $\bigvee_\mu x_\alpha^\mu = x_\alpha \in (m)_\alpha^-$, with $\phi_{\alpha\beta}x_\alpha = \bigvee_\mu \phi_{\alpha\beta}x_\alpha^\mu = \bigvee_\mu x_\beta^\mu = x_\beta$.

THEOREM 5. Let $B^d = \lim_{\rightarrow} B_\alpha$, and let the $\phi_{\alpha\beta}$ be onto mappings. Then the Boolean algebras B_α/J_α , for fixed ideal J in B^d and suitable induced mappings $\phi_{\alpha\beta}^-$, form a direct system, the direct limit of which is isomorphic to B^d/J .

PROOF. For $\alpha \leq \beta$, $x_\alpha^- \in B_\alpha/J_\alpha$, choose an antecedent $x_\alpha \in B_\alpha$ of x_α^- . Let $\phi_{\alpha\beta}^-x_\alpha^-$ be the coset of J_β in B_β with representative $\phi_{\alpha\beta}x_\alpha$. The mapping $\phi_{\alpha\beta}^-$ of B_α/J_α into B_β/J_β clearly does not depend upon the

representatives employed and is onto B_β/J_β . For $\alpha \leq \beta \leq \gamma$, $\phi_{\alpha\gamma}x_\alpha^- = (\phi_{\alpha\gamma}x_\alpha)^- = (\phi_{\beta\gamma}\phi_{\alpha\beta}x_\alpha)^- = \phi_{\beta\gamma}^-(\phi_{\alpha\beta}x_\alpha)^- = \phi_{\beta\gamma}^-\phi_{\alpha\beta}^-x_\alpha^-$. The direct system of the B_α/J_α thus exists, and its direct limit will be denoted by D . For $x = \{(\alpha, x_\alpha^-)\} \in D$, and for given $\alpha, \beta \in A$, choose antecedents x_α, x_β , where $(\beta, x_\beta^-) \sim (\alpha, x_\alpha^-)$. There exists $\gamma \geq \alpha, \beta$ such that $\phi_{\alpha\gamma}x_\alpha^- = \phi_{\beta\gamma}x_\beta^-$. That is, $(\phi_{\alpha\gamma}x_\alpha)^- = (\phi_{\beta\gamma}x_\beta)^-$. Computing the symmetric difference in $B^d = \lim_{\rightarrow} B_\alpha$, we have $\{(\alpha, x_\alpha)\} - \{(\beta, x_\beta)\} = \{(\gamma, \phi_{\alpha\gamma}x_\alpha - \phi_{\beta\gamma}x_\beta)\}$. But $\phi_{\alpha\gamma}x_\alpha - \phi_{\beta\gamma}x_\beta \in J_\gamma$, since $(\phi_{\alpha\gamma}x_\alpha)^- = (\phi_{\beta\gamma}x_\beta)^-$. Hence the symmetric difference in B^d is in J . There is thus defined a single-valued map Φ on D into B^d/J given by $\Phi\{(\alpha, x_\alpha^-)\} = \{(\alpha, x_\alpha)\}^-$. Φ is onto, since for each $\{(\alpha, x_\alpha)\}^- \in B^d/J$, $\Phi\{(\alpha, x_\alpha^-)\} = \{(\alpha, x_\alpha)\}^-$. If $\Phi\{(\alpha, x_\alpha^-)\} = \Phi\{(\beta, y_\beta^-)\}$, then $\{(\alpha, x_\alpha)\}^- = \{(\beta, y_\beta)\}^-$, and there exists $\gamma \geq \alpha, \beta$ with $\phi_{\alpha\gamma}x_\alpha - \phi_{\beta\gamma}y_\beta \in J_\gamma$. Thus, $\{(\alpha, x_\alpha^-)\} = \{(\gamma, \phi_{\alpha\gamma}x_\alpha^-)\} = \{(\gamma, (\phi_{\alpha\gamma}x_\alpha)^-)\} = \{(\gamma, (\phi_{\beta\gamma}y_\beta)^-)\} = \{(\gamma, \phi_{\beta\gamma}y_\beta^-)\} = \{(\beta, y_\beta^-)\}$. This shows that Φ^{-1} is single-valued, hence one-to-one on D onto B^d/J .

If $\{(\alpha, x_\alpha)\}^- \geq \{(\beta, y_\beta)\}^- = \{(\alpha, y_\alpha)\}^-$ in B^d/J , there exist $\{(\alpha, x_\alpha)\} \in \{(\alpha, x_\alpha)\}^-$, $\{(\alpha, y_\alpha)\} \in \{(\alpha, y_\alpha)\}^-$ with $\{(\alpha, x_\alpha)\} \geq \{(\alpha, y_\alpha)\}$ in B^d . There exists $\alpha \in A$ with $x_\alpha \geq y_\alpha$. Hence $x_\alpha^- \geq y_\alpha^-$, and $\{(\alpha, x_\alpha^-)\} \geq \{(\alpha, y_\alpha^-)\}$. Φ^{-1} thus preserves $\geq$. As for Φ , $\{(\alpha, x_\alpha^-)\} \geq \{(\alpha, y_\alpha^-)\} = \{(\beta, y_\beta^-)\}$ in D implies the existence of α such that $x_\alpha^- \geq y_\alpha^-$ in B_α/J_α . There exist $x_\alpha \in B_\alpha, y_\alpha \in B_\alpha$ with $x_\alpha \geq y_\alpha$; and $\{(\alpha, x_\alpha)\} \geq \{(\alpha, y_\alpha)\}$ in B^d . Also, $\{(\alpha, x_\alpha)\}^- \geq \{(\alpha, y_\alpha)\}^-$ in B^d/J . Hence $\Phi\{(\alpha, x_\alpha^-)\} \geq \Phi\{(\beta, y_\beta^-)\}$. This proves the theorem.

7. Ideal decomposition. To each $x_\alpha \in B_\alpha$ let correspond $\{(\alpha, x_\alpha)\} \in B^d$. This determines a mapping ϕ_α of B_α into B^d , a lattice homomorphism with kernel $\mathfrak{k}_\alpha$. ϕ_α is onto B^d ; for, $\{(\beta, x_\beta)\} \in B^d, \gamma \geq \alpha, \beta$ imply the existence of $x_\alpha \in B_\alpha$ with $\phi_{\alpha\gamma}x_\alpha = \phi_{\beta\gamma}x_\beta$, so that $\{(\alpha, x_\alpha)\} = \{(\gamma, \phi_{\beta\gamma}x_\beta)\} = \{(\beta, x_\beta)\}$.

Let J and K be ideals in a Boolean algebra B . Suppose that (1) $x \in J, K$ imply $x = 0$ and that (2) to each $x \in B$, there exist $y \in J, z \in K$ with $x = y \cup z$. It is easily seen that J and K are principal ideals. B is said to be the cardinal product [1] of J and K . In general, let Σ be an index class, and let $\{J^\sigma\}, \sigma \in \Sigma$, be a set of ideals of B , indexed by Σ , with the properties (1) for $x^\sigma \in J^\sigma$ and for every existing $\bigvee_{\tau \neq \sigma} x^\tau, x^\tau \in J^\tau, \tau \neq \sigma$, one has $x^\sigma \cap \bigvee_{\tau \neq \sigma} x^\tau = 0$; (2) to each $x \in B$ correspond $x^\sigma \in J^\sigma, \sigma \in \Sigma$, such that $x = \bigvee_{\sigma} x^\sigma$. B is said to be the cardinal product of the (principal) ideals J^σ .

THEOREM 6. Let the $\phi_{\alpha\beta}$ be onto mappings, where $B^d = \lim_{\rightarrow} B_\alpha$. B^d is the cardinal product of ideals J and K if, and only if, (a) for each $\alpha \in A$ and each $x_\alpha \in B_\alpha$, there exist $y_\alpha \in J_\alpha, z_\alpha \in K_\alpha$, and $k_\alpha \in \mathfrak{k}_\alpha$ with $x_\alpha - (y_\alpha \cup z_\alpha) = k_\alpha$; and (b) $x_\alpha \in J_\alpha, K_\alpha$ imply $x_\alpha \in \mathfrak{k}_\alpha$, for every $\alpha \in A$.

PROOF. If $B^d = JK$, then $\{(\alpha, x_\alpha)\} = \{(\alpha, y_\alpha)\} \cup \{(\alpha, z_\alpha)\}$, $y_\alpha \in J_\alpha$, $z_\alpha \in K_\alpha$. $x_\alpha - (y_\alpha \cup z_\alpha) \in \mathfrak{f}_\alpha$. As for (b), $x_\alpha \in J_\alpha$, K_α imply $\{(\alpha, x_\alpha)\} \in J, K$. Hence $\{(\alpha, x_\alpha)\} = 0$, and $x_\alpha \in \mathfrak{f}_\alpha$. Conversely, given (a) and (b), for $\{(\alpha, x_\alpha)\} \in B^d$, there exist $y_\alpha \in J_\alpha$, $z_\alpha \in K_\alpha$, and $k_\alpha \in \mathfrak{f}_\alpha$ with $x_\alpha - (y_\alpha \cup z_\alpha) = k_\alpha$. This implies $\{(\alpha, x_\alpha)\} = \{(\alpha, y_\alpha)\} \cup \{(\alpha, z_\alpha)\}$ so that J and K generate B^d . If $\{(\alpha, x_\alpha)\} \in J, K$, then $x_\alpha \in J_\alpha, K_\alpha$ so that $x_\alpha \in \mathfrak{f}_\alpha$ and $\{(\alpha, x_\alpha)\} = 0$.

COROLLARY. Under the conditions of the theorem, if each B_α is the cardinal product of J_α and K_α , then B^d is the cardinal product of J and K .

THEOREM 7. Let $B^d = \lim_{\leftarrow} B_\alpha$ where the B_α are complete. Let the $\phi_{\alpha\beta}$ be onto mappings. Suppose that B^d has the property that for every α , $\bigvee_\sigma \{(\alpha, x_\alpha^\sigma)\}$ exists and equals $\{(\alpha, \bigvee_\sigma x_\alpha^\sigma)\}$. Then B^d is the cardinal product of ideals $\{J^\sigma\}$ if, and only if, (a') for each $\alpha \in A$ and $x_\alpha \in B_\alpha$, there exist $y_\alpha^\sigma \in J_\alpha^\sigma$ and $k_\alpha \in \mathfrak{f}_\alpha$ with $x_\alpha - \bigvee_\sigma y_\alpha^\sigma = k_\alpha$; and (b') $x_\alpha^\sigma \in J_\alpha^\sigma$, $x_\alpha^\tau \in J_\alpha^\tau$, $\tau \neq \sigma$, yield $x_\alpha^\sigma \cap \bigvee_{\tau \neq \sigma} x_\alpha^\tau \in \mathfrak{f}_\alpha$.

The proof is like that of Theorem 6.

COROLLARY. If, under the conditions of Theorem 7, each B_α is the cardinal product of the J_α^σ , then B^d is the cardinal product of the J^σ .

8. **Ideals in the inverse limit.** For the inverse limits, the results of §§6 and 7 have appropriate counterparts. Since the proofs are straightforward or are similar to those of §§6 and 7, only the results will be stated. Given $B^i = \lim_{\leftarrow} (B_\alpha, \Pi_{\beta\alpha})$, the set of all $x_\alpha \in B_\alpha$ for which there exists $x \in B^i$ with α -component x_α is a subalgebra of B_α . $\Pi_{\beta\alpha}$ restricted to this subalgebra is onto the corresponding subalgebra of B_β ; and if we replace each B_α by its corresponding subalgebra (as defined here), then B^i is not changed. For the rest of this section, we assume this replacement to have been made.

For an ideal J in B^i , let J_α be the set of all $x_\alpha \in B_\alpha$ for which x_α is the α -component of some $x \in J$. Then J_α is an ideal in B_α . $\Pi_{\beta\alpha} J_\alpha = J_\beta$, and if the kernel of $\Pi_{\beta\alpha}$ is included in J_α , then $\Pi_{\beta\alpha}^{-1} J_\beta = J_\alpha$. Corresponding to each ideal J in B^i , there is an ideal $J^\sharp = \lim_{\leftarrow} (J_\alpha, \Pi_{\beta\alpha})$ in B^i such that $J \subset J^\sharp$ and such that $J_\alpha^\sharp = J_\alpha$ for every $\alpha \in A$. If J is prime, then $J^\sharp = J$ or $J^\sharp = B^i$. If some J_α is proper for J prime, then $J = J^\sharp$. If J is prime then J_α is prime or $J_\alpha = B_\alpha$ for a given index α . Likewise, if J is principal, then $J^\sharp = J$.

Let xJ represent that coset of J which contains the element x . For J an ideal in B^i , define a mapping $\Pi_{\beta\alpha}^*$ of B_α/J_α into B_β/J_β ($\beta \leq \alpha$) by $\Pi_{\beta\alpha}^*(x_\alpha J_\alpha) = (\Pi_{\beta\alpha} x_\alpha) J_\beta$. $\Pi_{\beta\alpha}^*$ turns out to be a homomorphism of B_α/J_α onto B_β/J_β and is independent of the representa-

tive x_α of the coset $x_\alpha J_\alpha$. If $J^\# = J$, then B^i/J is isomorphic to a subalgebra of $\lim_{\leftarrow} (B_\alpha/J_\alpha, \Pi_{\beta\alpha}^*)$.

$B^i = J^\# K^\#$, the cardinal product of $J^\#$ and $K^\#$, if, and only if, $B_\alpha = J_\alpha K_\alpha$ for every $\alpha \in A$. $B^i = JK$ implies the same statement as above on the B_α 's. If the B_α are complete and if the $\Pi_{\beta\alpha}$ are complete mappings, then B^i is the cardinal product of a collection of ideals $\{(J^\sigma)^\#\}$, a collection indexed by a set Σ , if and only if each B_α is the cardinal product of the collection $\{J_\alpha^\sigma\}$. If B^i is the cardinal product of the collection $\{J^\sigma\}$, then the same statement as above on the B_α 's holds.

9.2 The Stone spaces. Let B_α be a Boolean algebra, and let Ω_α be the collection of prime ideals of B_α . Ω_α is topologized by letting the neighborhoods of a base be all $\mathfrak{D}(x)$, where $\mathfrak{D}(x)$ is the set of all prime ideals of B_α which contain the element x of B_α . (See, for instance, [4].) Let $T_{\beta\alpha}$ be a homomorphism on a Boolean algebra B_α onto a subalgebra of a Boolean algebra B_β . For any prime ideal M_β in B_β , the complete inverse image $T_{\beta\alpha}^{-1}M_\beta$ is a prime ideal in B_α . Define a mapping $T_{\alpha\beta}^*$ on the Stone space Ω_β into the Stone space Ω_α by $T_{\alpha\beta}^*M_\beta = T_{\beta\alpha}^{-1}M_\beta$. For $x \in B_\alpha$, let $T_{\beta\alpha}x = y \in B_\beta$. A brief check shows that $\mathfrak{D}(y) = T_{\alpha\beta}^{*-1}\mathfrak{D}(x)$, where $\mathfrak{D}(x)$ is a base neighborhood in Ω_α , $\mathfrak{D}(y)$ in Ω_β . $T_{\alpha\beta}^*$ is thus continuous in a strong sense. We note that $x_\alpha \in T_{\alpha\beta}^*M_\beta$ if, and only if, $T_{\beta\alpha}x_\alpha \in M_\beta$.

Now let $R_{\alpha\beta}$ be a function continuous-in-the-strong-sense from Ω_β into Ω_α . That is, to $\mathfrak{D}(x)$, $x \in B_\alpha$, corresponds $\mathfrak{D}(y)$, $y \in B_\beta$, with $\mathfrak{D}(y) = R_{\alpha\beta}^{-1}\mathfrak{D}(x)$. Define a mapping $R_{\beta\alpha}^\#$ on B_α into B_β as follows. For $x \neq 0_\alpha$, $x \in B_\alpha$, note that $R_{\alpha\beta}^{-1}[\mathfrak{D}(x)]$ is a neighborhood $\mathfrak{D}(y')$ in Ω_β , $y' \in B_\beta$. Let $R_{\beta\alpha}^\#x = y$. For $x = 0_\alpha$, let $R_{\beta\alpha}^\#0_\alpha = 0_\beta$. Using Stone's representation theorem [3], it is easy to show that $R_{\beta\alpha}^\#$ is a homomorphism on B_α onto a subalgebra of B_β . Also, $x_\alpha \in R_{\alpha\beta}M_\beta$ if, and only if, $R_{\beta\alpha}^\#x_\alpha \in M_\beta$. Here, M_β is a prime ideal of B_β .

From these definitions, we can show that $T_{\beta\alpha}^{**} = T_{\beta\alpha}$ and that $R_{\alpha\beta}^{**} = R_{\alpha\beta}$. It follows that the homomorphisms of B_α onto subalgebras of B_β are in one-to-one correspondence with the continuous-in-the-strong-sense mappings of Ω_β into Ω_α . By standard arguments, we can show that $T_{\alpha\beta}^*$ is onto Ω_α if, and only if, $T_{\beta\alpha}$ is an isomorphism of B_α onto a subalgebra of B_β ; that if $T_{\beta\alpha}$ is onto B_β , then $T_{\alpha\beta}^*$ is one-to-one on Ω_β onto a subset of Ω_α ; and that $T_{\alpha\beta}^*$ is one-to-one on Ω_β onto a subset of Ω_α if, and only if, every prime ideal of the subalgebra $T_{\beta\alpha}B_\alpha$ can be extended to a prime ideal of B_β in one and only one

² Some of the statements of this section are due (with occasional modifications) to the referee, who suggested the investigation of this and of succeeding sections.

way. (This extension is the smallest convex set in B_β which contains the given prime ideal of $T_{\beta\alpha}B_\alpha$.)

10. The induced limits. Let B_α , B_β , and B_γ be Boolean algebras with appropriate homomorphisms $T_{\gamma\alpha}$ (on B_α onto a subalgebra of B_γ), and so forth. It follows quickly from the definitions that $T_{\gamma\alpha} = T_{\gamma\beta}T_{\beta\alpha}$ if, and only if, $T_{\alpha\gamma}^* = T_{\alpha\beta}^*T_{\beta\gamma}^*$.

This shows that to each inverse limit $\lim_{\leftarrow} (B_\alpha, \Pi_{\beta\alpha})$ of Boolean algebras there corresponds a direct limit $\lim_{\rightarrow} (\Omega_\alpha, \Pi_{\alpha\beta}^*)$ of the Stone spaces, where $\Pi_{\alpha\beta}^*$ is defined as in §9; and to each direct limit $\lim_{\rightarrow} (B_\alpha, \phi_{\alpha\beta})$ of Boolean algebras there corresponds an inverse limit $\lim_{\leftarrow} (\Omega_\alpha, \phi_{\beta\alpha}^*)$. Conversely, if the $\theta_{\beta\alpha}$ are continuous-in-the-strong-sense, then $\lim_{\leftarrow} (\Omega_\alpha, \theta_{\beta\alpha})$ induces $\lim_{\rightarrow} (B_\alpha, \theta_{\alpha\beta}^\#)$, and likewise for $\lim_{\rightarrow} (\Omega_\alpha, \psi_{\alpha\beta})$ and $\lim_{\leftarrow} (B_\alpha, \psi_{\beta\alpha}^\#)$, where the $\psi_{\alpha\beta}$ are continuous in the strong sense.

THEOREM 8. *Let S be the Stone space of $B^i = \lim_{\leftarrow} (B_\alpha, \Pi_{\beta\alpha})$, where each $x_\alpha \in B_\alpha$ is the α -component of some $x \in B^i$. Let $T = \lim_{\rightarrow} (\Omega_\alpha, \Pi_{\alpha\beta}^*)$ (where Ω_α is the Stone space of B_α). Then there exists a one-to-one map σ on T onto a closed subset of S .*

PROOF. By §8, all $\Pi_{\beta\alpha}$ are onto mappings. By §9, this makes the $\Pi_{\alpha\beta}^*$ one-to-one maps. Let y be an element of T . For an index α and for prime ideals M_α^1 and M_α^2 in B_α suppose that (α, M_α^1) and (α, M_α^2) are both in y . Then there exists an index $\beta \geq \alpha$ such that $\Pi_{\beta\alpha}^* M_\alpha^1 = \Pi_{\beta\alpha}^* M_\alpha^2$. Since $\Pi_{\beta\alpha}^*$ is one-to-one, $M_\alpha^1 = M_\alpha^2$. Now define a subset $N(y)$ of B^i by $N(y) = E[x; x \in B^i, (\alpha, M_\alpha) \in y \text{ implies } x_\alpha \in M_\alpha]$. 0, the infimum of B^i , is in $N(y)$ so that $N(y)$ is nonvoid. It is trivial to show that $N(y)$ is a prime ideal in B^i . Define a mapping σ on T into S by $\sigma(y) = N(y)$. By §8, $[N(y)]_\alpha = M_\alpha$.

Suppose that $\sigma(y) = \sigma(z) = N$, $y, z \in T$, $N \in S$. There is an index α and a pair of prime ideals, M_α^1, M_α^2 , in B_α such that $(\alpha, M_\alpha^1) \in y$, $(\alpha, M_\alpha^2) \in z$. But by the above, $M_\alpha^1 = N_\alpha$ and $M_\alpha^2 = N_\alpha$. Hence $y = z$, and σ is one-to-one.

Let M be a prime ideal of B^i . Suppose that $x \in M$ implies $x_\alpha = I_\alpha$ for no index $\alpha \in A$. By §8, each M_α is a prime ideal in B_α . For $\alpha \leq \beta$, $\Pi_{\beta\alpha}^* M_\alpha = M_\beta$, so that $y = \{(\alpha, M_\alpha)\}$ is a point of T . $\sigma(y) = N(y)$ where $[N(y)]_\alpha = M_\alpha$. Since $N(y)$ is prime in B^i , §8 shows that $N(y) = M$. We have thus determined the range of σ .

The complement of this range in S consists of all those prime ideals of B^i which contain elements of the form $x = \{x_\alpha\}$ where some $x_\alpha = I_\alpha$. Thus this complement is the set sum of open sets, ergo an open set; and the range of σ is closed in S .

COROLLARY 1. *Under the conditions of the theorem, let each $\Pi_{\beta\alpha}$ be an isomorphism of B_α onto B_β . Then range $\sigma = S$.*

The proof is immediate.

COROLLARY 2. *Under the conditions of the theorem, $T = \lim_{\leftarrow} (\Omega_\alpha, \Pi_{\alpha\beta}^*)$ can be given the topology of a Stone space.*

PROOF. σ is one-to-one on T into a closed, hence compact, subset C of S . C has a relative topology from S . Since S is totally disconnected, so is C in the relative topology. Hence C is a Stone space. Introducing the topology of C into T via σ , we make T a Stone space.

We should note that no a priori topology is to be expected in $T = \lim_{\leftarrow} (\Omega_\alpha, \Pi_{\alpha\beta}^*)$, even though each Stone space Ω_α has a topological structure and each $\Pi_{\alpha\beta}^*$ is continuous in-the-strong-sense. (See Hurewicz and Wallman, *Dimension theory*, Princeton, 1941, p. 133, footnote †.)

THEOREM 9. *Let S be the Stone space of $B^d = \lim_{\leftarrow} (B_\alpha, \phi_{\alpha\beta})$, and let $T = \lim_{\leftarrow} (\Omega_\alpha, \phi_{\alpha\beta}^*)$. Then S and T are homeomorphic.*

PROOF. A set of prime ideals $y = \{M_\alpha\}$, indexed by A , is an element of T if, and only if, $M_\alpha \in \Omega_\alpha$, and $\alpha \leq \beta$ implies $\phi_{\beta\alpha}^* M_\beta = M_\alpha$ (that is, $\phi_{\alpha\beta}^{-1} M_\beta = M_\alpha$). Let $N(y) = E[x; x \in B^d, x = \{(\alpha, x_\alpha)\}$ where each such $x_\alpha \in M_\alpha$]. A tedious but trivial proof shows that $N(y)$ is a prime ideal in B^d . Also the α -component $[N(y)]_\alpha$ of $N(y)$ turns out to be M_α .

Now let λ be a mapping on T into S given by $\lambda(y) = N(y)$. If $\lambda\{M_\alpha^{(1)}\} = \lambda\{M_\alpha^{(2)}\} = N$, then $M_\alpha^{(1)} = N_\alpha = M_\alpha^{(2)}$ for every $\alpha \in A$, and λ is one-to-one. For a given prime ideal N of B^d it is immediate that $\alpha \leq \beta$ implies $\phi_{\alpha\beta}^{-1} N_\beta = N_\alpha$, so that $y = \{N_\alpha\} \in T$. A brief check shows that $\lambda(y) = N$. Hence λ is onto S .

In the Cartesian product $P_\beta \Omega_\beta$ of all the Stone spaces Ω_β , we can pick out a basis for the standard topology as follows. We select a finite set of indices $\alpha(1), \alpha(2), \dots, \alpha(n) \in A$. Choose elements $x_{\alpha(1)}^{(1)} \in B_{\alpha(1)}$, $x_{\alpha(2)}^{(2)} \in B_{\alpha(2)}$, $\dots$, $x_{\alpha(n)}^{(n)} \in B_{\alpha(n)}$. Then form the Cartesian product $\mathfrak{D}^*$ of factors Ω_α for $\alpha \in A$, $\alpha \neq \alpha(i)$, $\mathfrak{D}(x_{\alpha(i)}^{(i)})$, $i = 1, 2, \dots, n$. (See [2, pp. 6-10].) Let $\mathfrak{D}$ be the relativization of $\mathfrak{D}^*$ in T . The set of all such $\mathfrak{D}$ forms a basis for the relative topology of T in $P_\beta \Omega_\beta$. Set $x^{(i)} = \{(\alpha(i), x_{\alpha(i)}^{(i)})\} \in B^d$, $i = 1, 2, \dots, n$. Now $y = \{M_\alpha\} \in \mathfrak{D}$ if, and only if, $x^{(i)} \in N(y)$, $i = 1, 2, \dots, n$. For, if $\{M_\alpha\} \in \mathfrak{D}$ and if $\{(\alpha, x_\alpha)\} = \{(\alpha(i), x_{\alpha(i)}^{(i)})\}$, then there exists an index $\gamma \geq \alpha$, $\alpha(i)$ such that $\phi_{\alpha\gamma} z_\alpha = \phi_{\alpha(i)\gamma} x_{\alpha(i)}^{(i)} \in M_\gamma$, since $x_{\alpha(i)}^{(i)} \in M_{\alpha(i)}$. But $\phi_{\alpha\gamma}^{-1} M_\gamma = M_\alpha$. Hence $z_\alpha \in M_\alpha$. This shows that $x^{(i)}$ is in $N(y)$. The converse is obvi-

ous. The above statement is equivalent to the fact that $\lambda(\mathfrak{D}) = \mathfrak{D}(x^{(1)}) \cap \mathfrak{D}(x^{(2)}) \cap \cdots \cap \mathfrak{D}(x^{(n)})$, an open set in S . Thus λ^{-1} is continuous.

Conversely, for $x \in B^d$, let $\mathfrak{Q}$ be that subset of T for which $\lambda(\mathfrak{Q}) = \mathfrak{D}(x)$. Choose $\alpha \in A$ such that $x = \{(\alpha, x_\alpha)\}$. Then $y = \{M_\alpha\} \in \mathfrak{Q}$ if, and only if, $x_\alpha \in M_\alpha$. Hence $\mathfrak{Q}$ is the relativization of a Cartesian product of factors Ω_β if $\beta \neq \alpha$, and $\mathfrak{D}(x_\alpha)$. $\mathfrak{Q}$ is thus open, and λ is continuous. This proves that λ is indeed a homeomorphism.

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INTEGRAL BASES AND VARIETIES MULTIPLY OF THE FIRST SPECIES

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1. The object of this note is to point out the connection between varieties multiply of the first species in the sense of Dubreil¹ and those irreducible varieties V which are such that the ring of homogeneous coordinates of the general point of V admits an independent linear base over the subring generated by a suitably chosen set of independent variables.

2. Let k be a field of characteristic zero and let $R = k[X_0, X_1, \dots, X_n]$ be the ring of polynomials in $n+1$ indeterminates over k . If $\mathfrak{a}$ is an H -ideal in R with no irrelevant components, $\mathfrak{a}$ is said to be of the first species if there exists a linear form $L = \sum c_i X_i$, $c_i \in k$, which does not belong to any of the relevant prime ideals of $\mathfrak{a}$ and is such that $\mathfrak{a} + R \cdot L$ has no irrelevant components. It is known

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¹ P. Dubreil, *Quelques propriétés des variétés algébriques se rattachant aux théories de l'algèbre moderne*, Actualités Scientifiques et Industrielles, vol. 210, 1935. P. Dubreil, *Sur la dimension des idéaux de polynômes*, J. Math. Pures Appl. (9) vol. 15 (1936). These will be referred to as D1 and D2 respectively.