ON THE REPRESENTATION OF INTEGERS BY SUMS OF POLYNOMIALS

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All letters in this paper stand for integers.

L. K. Hua [1]¹ proved the following theorem.

THEOREM. *For any positive integer a , every integer can be expressed in infinitely many ways as a sum of seven values of the cubic function*

$$f(x) = \frac{a}{6}(x^3 - x) + x$$

for integral values of x .

A. Sugar [2] improved this result. He deduced the following theorem.

THEOREM. *For any a , c , and any g prime to a , every integer can be expressed in infinitely many ways as a sum of five values of the function*

$$p(x) = \frac{a}{6}(x^3 - x) + gx + c$$

for integral values of x .

The last theorem now is a special case ($k=1$) of the following theorem.

THEOREM I. *For any a , c , and g prime to a , and any $k \geq 1$, every integer can be expressed in infinitely many ways as a sum of (4^k+1) values of the (2^k+1) th degree polynomial in x*

$$\phi(x) = \frac{a(x+k)(x+k-1) \cdots (x-k)}{(2^k+1)!} + gx + c$$

for integral values of x .

Furthermore we deduce:

THEOREM II. *For any g and c , and any $k \geq 1$, every integer can be expressed as a sum of 4^k values of the (2^k+1) th degree polynomial in x*

$$\phi(x) = \frac{(x+k)(x+k-1) \cdots (x-k)}{(2^k+1)!} + gx + c$$

for integral values of x .

Received by the editors October 7, 1950.

¹ Numbers in brackets refer to the literature cited at the end of the paper.

If now n is an arbitrary integer, then, as $gt+c$ runs through a complete set of residues mod $|a|$ when t does, there exist infinitely many integers t , such that

$$n - 2^{2k}c \equiv gt + c \pmod{|a|},$$

so that from (1)

$$n - 2^{2k}c \equiv \psi(t, k) + gt + c \pmod{|a|},$$

or

$$n = \phi(t) + ma + 2^{2k}c = \phi(t) + S(m).$$

This completes the proof of Theorem I.

For $a=1$ the identity $\Delta_x^{2k}\phi(x-k)=xa$ becomes

$$x + 2^{2k}c = S(x)$$

and S runs through all integers when x does. This proves Theorem II.

LITERATURE

1. L. K. Hua, *On the representation of integers by sums of seven cubic functions*, Tôhoku Math. J. vol. 41 (1935-1936) pp. 361-366.
2. Alvin Sugar, *On a result of Hua for cubic polynomials*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 164-165.

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