LINEAR DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS

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Tomlinson Fort [1]¹ has discussed in detail, by using an ingenious idea, the nature of the solution of this type of equations of the first and second orders; and incidentally has shown that: *If*

$$y(i+2) + M(i)y(i+1) + y(i) = 0$$

and

$$M(i+\omega) = M(i),$$

then

$$y(i+\omega) + y(i-\omega) = 2Ay(i),$$

where A is a constant and i and ω are, of course, integers.

In other words "A second order equation with a periodic coefficient of period ω can be transformed into an equation of order 2ω with constant coefficients."

Here the generalization of this result for any order is proved by using an entirely different method of approach.

In what follows x is the independent integral variable and the period λ is naturally a positive integer.

THEOREM. *A linear difference equation of order n with periodic coefficients of common period λ can be transformed into a linear difference equation of order $n\lambda$ with constant coefficients.*

Or more precisely: If

$$(1) \quad \left\{ \sum_{i=0}^n a_i(x) E^i \right\} u(x) = 0,$$

$$a_i(x+\lambda) = a_i(x) \quad (i = 0, 1, \dots, n),$$

then there exist constants C_i ($i=0, 1, \dots, n$) such that

$$(2) \quad \left\{ \sum_{i=0}^n C_i E^{i\lambda} \right\} u(x) = 0.$$

PROOF. *Case (1) $\lambda \leq n$.* Without loss of generality we may assume that $a_n(x) = -1$. Then using an idea of Milne-Thompson [2] we may

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¹ Numbers in brackets refer to the references at the end of the paper.

write

$$\begin{aligned}
 \begin{bmatrix} u(x+n) \\ u(x+n-1) \\ u(x+n-2) \\ \dots \\ u(x+1) \end{bmatrix} &= \begin{bmatrix} a_{n-1}(x) & a_{n-2}(x) & \dots & a_1(x) & a_0(x) \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \\
 (3) \quad &\times \begin{bmatrix} u(x+n-1) \\ u(x+n-2) \\ u(x+n-3) \\ \dots \\ u(x) \end{bmatrix} \\
 &= [a_{rs}(x)] \begin{bmatrix} u(x+n-1) \\ u(x+n-2) \\ \dots \\ u(x) \end{bmatrix}.
 \end{aligned}$$

Equation (3) defines $[a_{rs}(x)]$ and shows that

$$(3.1) \quad [a_{rs}(x+\lambda)] = [a_{rs}(x)].$$

By successive applications of (3) we get

$$\begin{aligned}
 \begin{bmatrix} u(\lambda+x+n-1) \\ u(\lambda+x+n-2) \\ \dots \\ u(\lambda+x) \end{bmatrix} &= [a_{rs}(x+\lambda-1)] \times \dots \\
 (4) \quad &\times [a_{rs}(x+1)][a_{rs}(x)] \begin{bmatrix} u(x+n-1) \\ u(x+n-2) \\ \dots \\ u(x) \end{bmatrix} \\
 &= [b_{rs}(x)] \begin{bmatrix} u(x+n-1) \\ u(x+n-2) \\ \dots \\ u(x) \end{bmatrix}.
 \end{aligned}$$

Equation (4) defines $[b_{rs}(x)]$ and by (3.1) shows that

$$(4.1) \quad [b_{rs}(x + \lambda)] = [b_{rs}(x)].$$

By successive applications of (4) and (4.1) we get

$$(5) \quad \begin{bmatrix} u(i\lambda + x + n - 1) \\ u(i\lambda + x + n - 2) \\ \dots \\ u(i\lambda + x) \end{bmatrix} = [b_{rs}(x)]^i \begin{bmatrix} u(x + n - 1) \\ u(x + n - 2) \\ \dots \\ u(x) \end{bmatrix}.$$

Now let the n -dimensional vector v_i be defined by

$$(6) \quad v_i = \begin{bmatrix} u(i\lambda + x + n - 1) \\ u(i\lambda + x + n - 2) \\ \dots \\ u(i\lambda + x) \end{bmatrix}.$$

Let T be the homogeneous linear transformation whose matrix is

$$(7) \quad [b_{rs}(x)].$$

Then, since the $(n+1)$ th vector v_n is a linear function of the n vectors $v_0, v_1, \dots, v_{n-1}$ (assumed to be linearly independent), there exist $c_i(x)$ ($i=0, 1, \dots, n$) such that

$$(8) \quad \sum_{i=0}^n c_i(x) v_i = 0.$$

Then from equation (5) it follows that

$$(9) \quad \left\{ \sum_{i=0}^n c_i(x) T^i \right\} v_0 = 0.$$

From equation (9) it is clear that the functions $c_i(x)$ are functions of the coefficients of T^i ($i=1, 2, \dots, n$) and not of the components of the vectors. Therefore the functions $c_i(x)$ are functions of the elements of $[b_{rs}(x)]$ alone. Hence by (4.1)

$$(10) \quad c_i(x + \lambda) = c_i(x) \quad (i = 0, 1, \dots, n).$$

Equating the components of (8) separately to zero, we get

$$(11) \quad \sum_{i=0}^n c_i(x) u(i\lambda + x + j) = 0 \quad (j = 0, 1, \dots, n - 1).$$

Now put $x = k\lambda$, so that $c_i(x) = c_i(k\lambda) = c_i(0)$ by equation (10). The equation (11) becomes

$$\sum_{i=0}^n c_i(0)u(i\lambda + k\lambda + j) = 0 \quad (j = 0, 1, \dots, n-1).$$

But $\lambda \leq n$ and $j = 0, 1, \dots, (n-1)$, and so any x is of the form $k\lambda + j$. Hence the last equation becomes

$$(12) \quad \sum_{i=0}^n c_i(0)u(i\lambda + x) = 0$$

which proves the required result (2).

Case (2) $\lambda > n$. The proof is essentially the same, the primary modification being that equation (3) is replaced by

$$(13) \quad \begin{bmatrix} u(x + \lambda) \\ u(x + \lambda - 1) \\ u(x + \lambda - 2) \\ \dots \\ u(x + 1) \end{bmatrix} = \begin{bmatrix} a_{n-1}(x + \lambda - n) & a_{n-2}(x + \lambda - n) & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} \times \begin{bmatrix} u(x + \lambda - 1) \\ u(x + \lambda - 2) \\ u(x + \lambda - 3) \\ \dots \\ u(x) \end{bmatrix}$$

so that the matrix $[a_{rs}(x)]$ is now $(\lambda - 1) \times \lambda$ instead of $n \times n$.

Form of solution. Let

$$(14) \quad \omega = \cos \frac{2\pi}{\lambda} + (-1)^{1/2} \sin \frac{2\pi}{\lambda}.$$

If ρ^λ is a root, repeated p times, of the equation

$$\sum_{i=0}^n c_i z^i = 0,$$

then the corresponding roots of the auxiliary equation of equation (2) are

$$\rho, \rho\omega, \rho\omega^2, \dots, \rho\omega^{\lambda-1} \text{ each repeated } p \text{ times.}$$

The corresponding terms in the general solution of equation (2) are

$$\rho^x \left\{ \sum_{j=0}^{\lambda-1} \sum_{i=0}^{p-1} A_{ij} x^i \omega^{jx} \right\}.$$

Therefore the solution of equation (2) and hence that of (1) will contain terms of the form

$$(15) \quad \rho^x x^i \left(A \cos \frac{2\pi jx}{\lambda} + B \sin \frac{2\pi jx}{\lambda} \right)$$

where ρ , A , B are constants and i , j are integers.

But the solution of equation (2) will have $n\lambda$ arbitrary constants; and hence when this solution becomes a solution of equation (1) also there will be $n(\lambda-1)$ equations connecting the $n\lambda$ arbitrary constants.

REFERENCES

1. Tomlinson Fort, *Finite differences*, 1948, p. 205.
2. L. M. Milne-Thompson, *Finite differences*, 1933, p. 379.

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