A GENERALIZATION OF ABEL'S TRANSFORMATION

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In this paper we give a generalization of Abel's transformation and its application to infinite series. The generalization is contained in the first theorem. In Theorems 1 through 7 the a 's and b 's are arbitrary constants. Throughout the paper p will be an integer greater than one. We agree that all empty sums vanish.

THEOREM 1. *For every positive integral value of p*

$$(P) \quad \sum_{j=n+1}^{n+k} a_j^p b_j = \sum_{j=n+1}^{n+k} A_j^p (b_j - b_{j+1}) - A_n^p b_{n+1} + A_{n+k}^p b_{n+k+1} \\ - \sum_{i=0}^{p-2} C_{p,i+1} \sum_{j=n}^{n+k-1} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1},$$

where $A_k = \sum_{r=0}^k a_r$.

PROOF. The proof is by induction.

COROLLARY 1. *For $n = -1$ we have*

$$(Q) \quad \sum_{j=0}^{k-1} a_j^p b_j = \sum_{j=0}^{k-1} A_j^p (b_j - b_{j+1}) + A_{k-1}^p b_k \\ - \sum_{i=0}^{p-2} C_{p,i+1} \sum_{j=0}^{k-2} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1}.$$

COROLLARY 2. *For $p = 1$ we have Abel's transformation.*

THEOREM 2. *If*

$$(1) \quad \sum_{j=0}^{\infty} a_j^p \text{ converges to } S, \\ (2) \quad \sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} \text{ converge to } S_i \quad (i = 0, \dots, p-2),$$

then

$$A_n^p \text{ converges to } S + \sum_{i=0}^{p-2} C_{p,i+1} S_i.$$

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PROOF. If in (Q) we set $b_k \equiv 1$, we have

$$A_n^p = \sum_{j=0}^n a_j^p + \sum_{j=0}^{p-2} C_{p,i+1} \sum_{j=0}^{n-1} A_j^{p-1-i} a_{j+1}^{i+1},$$

from which the theorem follows in the limit.

COROLLARY 1. *If any p of the $p+1$ infinite series*

$$\sum_{j=0}^{\infty} a_j, \quad \sum_{j=0}^{\infty} a_j^p, \quad \sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} \quad (i = 0, \dots, p-2)$$

converge, the other does also.

THEOREM¹ 3. *If*

$$(1) \quad \sum_{j=0}^{\infty} A_j^p (b_j - b_{j+1}) \text{ converges,} \quad (2) \quad \lim_{n \rightarrow \infty} A_n^p b_{n+1} \text{ exists,}$$

$$(3) \quad \sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1} \text{ converge} \quad (i = 0, \dots, p-2),$$

then

$$\sum_{j=0}^{\infty} a_j^p b_j \text{ converges.}$$

PROOF. The theorem follows directly from (Q).

THEOREM 4. *If*

$$(1) \quad \sum_{j=0}^{\infty} a_j \text{ converges,}$$

$$(2) \quad (b_n) \text{ is monotonic and bounded,}$$

$$(3) \quad \sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1} \text{ converge} \quad (i = 0, \dots, p-2),$$

then

$$\sum_{j=0}^{\infty} a_j^p b_j \text{ converges.}$$

PROOF. Since (A_n) and (b_n) converge, so also does $(A_n^p b_{n+1})$. Since $\lim_{n \rightarrow \infty} b_n$ exists and $\sum_{j=0}^p (b_j - b_{j+1}) = b_0 - b_{n+1}$, $\sum_{j=0}^{\infty} (b_j - b_{j+1})$

¹ Theorems 3, 4, 5, 6, 7, 9, and 10 generalize, in a sense, known theorems. See Knopp, *Theory and applications of infinite series*, London, Blackie, 1948, pp. 314-315, 346-348.

converges, and converges absolutely because (b_n) is monotonic. Since (A_n) is a bounded sequence, so also is (A_n^p) and thus $\sum_{j=0}^{\infty} A_j^p(b_j - b_{j+1})$ converges. The theorem then follows from Theorem 3.

THEOREM 5. *If*

- (1) $\sum_{j=0}^{\infty} a_j$ has bounded partial sums,
- (2) (b_n) is monotonic and $\lim_{n \rightarrow \infty} b_n = 0$,
- (3) $\sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1}$ converge $(i = 0, \dots, p-2)$,

then

$$\sum_{j=0}^{\infty} a_j^p b_j \text{ converges.}$$

PROOF. Hypotheses (1) and (2) are sufficient to establish (1) and (2) of Theorem 3, from which the theorem follows.

THEOREM 6. *If*

- (1) $\sum_{j=0}^{\infty} a_j$ converges,
- (2) $\sum_{j=0}^{\infty} (b_j - b_{j+1})$ converges absolutely,
- (3) $\sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1}$ converge $(i = 0, \dots, p-2)$,

then

$$\sum_{j=0}^{\infty} a_j^p b_j \text{ converges.}$$

PROOF. Since (A_n) is bounded, $\sum_{j=0}^{\infty} A_j^p(b_j - b_{j+1})$ converges. Since $\sum_{j=0}^n (b_j - b_{j+1}) = b_0 - b_{n+1}$, we have that $\lim_{n \rightarrow \infty} b_n$ exists, and so from (1) $\lim_{n \rightarrow \infty} A_n^p b_{n+1}$ exists. Thus the theorem follows from Theorem 3.

THEOREM 7. *If*

- (1) $\sum_{j=0}^{\infty} a_j$ has bounded partial sums,

$$(2) \quad \sum_{j=0}^{\infty} (b_j - b_{j+1}) \text{ converges absolutely,}$$

$$(3) \quad \lim_{n \rightarrow \infty} b_n = 0,$$

$$(4) \quad \sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1} \text{ converge} \quad (i = 0, \dots, p-2),$$

then

$$\sum_{j=0}^{\infty} a_j^p b_j \text{ converges.}$$

PROOF. From (1) and (2), $\sum_{j=0}^{\infty} A_j^p (b_j - b_{j+1})$ converges. From (1) and (3), $\lim_{n \rightarrow \infty} A_n^p b_{n+1} = 0$, so the theorem follows from Theorem 3.

In the theorems that follow $a_j = a_j(x)$, $b_j = b_j(x)$ are functions of x defined in the interval $I = [a, b]$. The formulas corresponding to (P) and (Q) are denoted by (P') and (Q').

THEOREM 8. If

$$(1) \quad \sum_{j=0}^{\infty} a_j^p \text{ converges uniformly in } I,$$

$$(2) \quad \sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} \text{ converge uniformly in } I \quad (i = 0, \dots, p-2),$$

then

$$A_n^p \text{ converges uniformly in } I.$$

PROOF. From (P') if we set $b_j(x) \equiv 1$ we get

$$A_{n+k}^p - A_n^p = \sum_{j=n+1}^{n+k} a_j^p + \sum_{i=0}^{p-2} C_{p,i+1} \sum_{j=n}^{n+k-1} A_j^{p-1-i} a_{j+1}^{i+1}.$$

Given $\epsilon > 0$, we have from (1), if $n > n_1$,

$$\left| \sum_{j=n+1}^{n+k} a_j^p \right| < \frac{\epsilon}{2},$$

and from (2), if $n > n_2$,

$$\left| \sum_{j=n}^{n+k-1} A_j^{p-1-i} a_{j+1}^{i+1} \right| < \frac{\epsilon}{2 \sum_{i=0}^{p-2} C_{p,i+1}} \quad (i = 0, \dots, p-2).$$

Thus if $n > \max(n_1, n_2)$, we have $|A_{n+k}^p - A_n^p| < \epsilon$, from which the theorem follows.

THEOREM 9. *If*

- (1) $\sum_{i=0}^{\infty} A_i^p(b_i - b_{i+1})$ converges uniformly in I ,
- (2) $(A_n^p b_{n+1})$ converges uniformly in I ,
- (3) $\sum_{i=0}^{\infty} A_i^{p-1-i} a_{i+1}^{i+1} b_{i+1}$ converge uniformly in I ($i = 0, \dots, p-2$),

then

$$\sum_{i=0}^{\infty} a_i^p b_i \text{ converges uniformly in } I.$$

PROOF. Given $\epsilon > 0$, we have from (2), for $n > n_1$,

$$|A_{n+k}^p b_{n+k+1} - A_n^p b_{n+1}| < \frac{\epsilon}{3}.$$

For $n > n_2$ we have from (1)

$$\left| \sum_{i=n+1}^{n+k} A_i^p(b_i - b_{i+1}) \right| < \frac{\epsilon}{3}.$$

For $n > n_3$ we have from (3)

$$\left| \sum_{i=n}^{n+k-1} A_i^{p-1-i} a_{i+1}^{i+1} b_{i+1} \right| < \frac{\epsilon}{3 \sum_{i=0}^{p-2} C_{p, i+1}} \quad (i = 0, \dots, p-2).$$

Thus if $n > \max(n_1, n_2, n_3)$, we have from (P')

$$\left| \sum_{i=n+1}^{n+k} a_i^p b_i \right| < \epsilon,$$

from which the theorem follows.

THEOREM 10. *If*

- (1) $\sum_{i=0}^{\infty} a_i$ has uniformly bounded partial sums in I ,
- (2) (b_n) converges uniformly to zero in I ,

(3) *for fixed x the convergence of (b_n) is monotonic,*

(4) $\sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1}$ *converge uniformly in I ($i = 0, \dots, p-2$),*

then

$\sum_{j=0}^{\infty} a_j^p b_j$ *converges uniformly in I .*

PROOF. From (1) and (2) we have that $(A_n^p b_{n+1})$ converges uniformly in I . If K is greater than $|A_n(x)|$ for every x in I , we have for fixed x , from (3),

$$\begin{aligned} \left| \sum_{j=n+1}^{n+k} A_j^p (b_j - b_{j+1}) \right| &\leq K^p \sum_{j=n+1}^{n+k} |b_j - b_{j+1}| \\ &= K^p \left| \sum_{j=n+1}^{n+k} (b_j - b_{j+1}) \right| \\ &= K^p |b_{n+1} - b_{n+k+1}|, \end{aligned}$$

which tends to zero by (2). Thus the theorem follows from Theorem 9.

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