ORTHOGONAL SIMILARITY IN INFINITE-DIMENSIONAL SPACES¹

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1. **Introduction.** In [2]² the author initiated the study, in a purely algebraic way, of infinite-dimensional vector spaces with an inner product. The two main topics were the exhibition of canonical forms for the inner product, and the classification of subspaces relative to the inner product. In this paper we take up a third topic: the classification of linear transformations relative to the inner product. This problem does not appear to have an easy solution, even when we make the following five assumptions: (1) the vector space has countable dimension, (2) the base field is algebraically closed, (3) the inner product is symmetric, (4) the linear transformation is self-adjoint, (5) the linear transformation is algebraic, that is, satisfies a polynomial equation.

Specifically, the problem is to give a complete set of "orthogonal" invariants for such a linear transformation T . In the finite-dimensional case (characteristic $\neq 2$) a classical theorem asserts that the invariants of T under similarity (in other words, the elementary divisors) already constitute a complete set of orthogonal invariants. We shall see that the theorem (at least in this form) does not generalize to the infinite-dimensional case. However we shall prove that it does generalize successfully in case T is homogeneous in the sense that all its elementary divisors have the same degree; and we shall give a complete discussion of the case $T^2 = 0$.

The questions treated in this paper do not seem to have been previously discussed to any great extent. It is to be observed that our assumptions are such as to preclude any overlapping with Hilbert space theory; nor is there any intersection with the interesting paper of Pontrjagin [4].

2. **The classical argument.** Let F be an algebraically closed field and V a vector space over F . We assume that V carries a nonsingular symmetric inner product. Let T be a linear transformation on V . The linear transformation T^* is said to be the *adjoint* of T if $(xT, y) = (x, yT^*)$ for all x and y in V . If an adjoint exists, it is unique, and it is appropriate to call T "continuous" if it has an adjoint. T is said

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² Numbers in brackets refer to the bibliography at the end of the paper.

to be *self-adjoint* if $T^* = T$, *skew* if $T^* = -T$. The following two statements are equivalent: (a) T is a nonsingular linear transformation preserving the inner product, (b) T has an adjoint T^* satisfying $TT^* = T^*T =$ the identity I . We call such a T *orthogonal*.

We now sketch a classical argument³ for solving the problem of orthogonal equivalence. Let A and B be linear transformations that are similar under a continuous linear transformation P :

$$(1) \quad P^{-1}AP = B.$$

If A and B are both self-adjoint, or both skew, or both orthogonal, we deduce from (1) that

$$(2) \quad P^{-1}A^*P = B^*.$$

From (1) and (2) one computes that PP^* commutes with A . Now, if PP^* is algebraic, and if the characteristic is different from two, there exists a polynomial f (with coefficients in F) such that $f(PP^*)$ is a square root R of PP^* . Then $R^* = R$, R also commutes with A , $U = R^{-1}P$ is orthogonal, and $U^{-1}AU = B$. In other words, under the stated conditions, similarity of A and B implies orthogonal similarity.

However, in the infinite-dimensional case, we have no assurance that PP^* is algebraic, even if it is assumed that both A and B are algebraic. Hence the classical argument seems to be virtually useless to us, and we must resort to other methods.

3. Reduction to the nilpotent case. Let again F be algebraically closed, V a vector space over F with a nonsingular symmetric inner product, and let T be a self-adjoint linear transformation on V . We assume further that T is locally algebraic: for any vector $x \in V$ there exists a polynomial f such that $xf(T) = 0$. For any scalar λ , define $V(\lambda)$ to be the subspace of vectors annihilated by some power of $T - \lambda I$ (the power depending on the vector). One knows that V is a direct sum, at least in the vector space sense, of the $V(\lambda)$. We now show that this is actually an orthogonal decomposition as well.

LEMMA 1. *If $\lambda \neq \mu$, the subspaces $V(\lambda)$ and $V(\mu)$ are orthogonal.*

PROOF. Let $x \in V(\lambda)$, $y \in V(\mu)$. Thus $x(T - \lambda I)^r = 0$, $y(T - \mu I)^s = 0$. Let u be an indeterminate. There exist polynomials g and h such that

$$g(u)(u - \lambda)^r + h(u)(u - \mu)^s = 1.$$

Let $z = yg(T)$ and we have $y = z(T - \lambda I)^r$. Then

³ For a complete account of this argument, see [1].

$$(x, y) = (x, z(T - \lambda I)^r) = (x(T - \lambda I)^r, z) = 0.$$

Thus to study T it will suffice to study it on one of the subspaces $V(\lambda)$. In other words, after replacing $T - \lambda I$ by T , we may assume that T is locally nilpotent, in the sense that every vector is annihilated by a suitable power of T .

We shall shortly assume that T is actually nilpotent, but before doing so we observe that T is a rather special kind of locally nilpotent linear transformation, for there are no vectors of infinite height (x has infinite height if x is in the range of T^n for every n). For if z is any vector, say $zT^k = 0$, then $x = yT^k$, and $(x, z) = (yT^k, z) = (y, zT^k) = 0$; hence $x = 0$. Let us now assume further that V has countable dimension. Then by a theorem of Prüfer,⁴ V is a direct sum of finite-dimensional invariant subspaces. Thus from the point of view of similarity, the structure of T is completely known. But orthogonal similarity is another matter.

4. The homogeneous case. Let T be a nilpotent linear transformation on a vector space V . It is known [5] that V can be expressed as a direct sum of finite-dimensional subspaces which are invariant under T and indecomposable. This decomposition is not unique in the absolute sense, but it is unique up to isomorphism; in other words, the (cardinal) number of subspaces of a given dimension i is an invariant. We refer to this invariant as the *multiplicity* of the elementary divisor λ^i .

These invariants constitute a complete set of invariants for T under similarity. However if it is orthogonal similarity that is being studied, then we need an orthogonal decomposition of V . The crucial question thus arises: can we express V as an *orthogonal* direct sum of finite-dimensional invariant subspaces? A tool for attacking this problem is given in the following lemma.

LEMMA 2. *Let V be a vector space with a nonsingular symmetric inner product, and let T be a self-adjoint linear transformation on V satisfying $T^n = 0$. Then: (a) if x is a vector satisfying $(xT^{n-1}, x) \neq 0$, the invariant subspace generated by x is nonsingular. (b) If x and y satisfy $(xT^{n-1}, x) = 0$, $(xT^{n-1}, y) \neq 0$, the invariant subspace generated by x and y is nonsingular. (c) If $xT^{n-1} \neq 0$, then x can be embedded in a finite-dimensional invariant nonsingular subspace.*

PROOF. (a) The invariant subspace S generated by x is spanned by

⁴ Cf. [5]. Prüfer proves the theorem for abelian groups, but the proof is valid without change for modules over any principal ideal ring, and in particular for linear transformations.

$x, xT, \dots, xT^{n-1}$. Since $T^n=0$ and T is self-adjoint, we have that (xT^i, xT^j) is 0 for $i+j \geq n$, and nonzero for $i+j=n-1$. Now suppose $z = \sum a_i xT^i$ annihilates all of S . On taking the inner product of z and $xT^{n-1}, xT^{n-2}, \dots$ we find in succession $a_0=0, a_1=0, \dots, z=0$.

(b) Suppose

$$z = \sum a_i xT^i + \sum b_j yT^j$$

annihilates each xT^i and yT^j ($i, j=0, \dots, n-1$). On taking inner products with $xT^{n-1}, yT^{n-1}, xT^{n-2}, yT^{n-2}, \dots$ we find in succession $b_0=0, a_0=0, b_1=0, a_1=0, \dots, z=0$.

(c) If $(xT^{n-1}, x) \neq 0$ we quote part (a). Suppose it is 0. By the nonsingularity of the inner product, there must exist y with $(xT^{n-1}, y) \neq 0$. We quote part (b).

Let us now see what information can be extracted from Lemma 2. First we recall that if T is self-adjoint on V , and S is a finite-dimensional nonsingular invariant subspace of V , then $V=S \oplus S'$ where S' is the orthogonal complement of S and is again invariant under T . Lemma 2 shows that we can tear off in this fashion successive finite-dimensional summands. However the process cannot be pursued by transfinite induction, since an infinite-dimensional nonsingular subspace is not necessarily a direct summand. To get the process to terminate we assume that V has countable dimension. Suppose we assume further that T is *homogeneous* in the sense that all its elementary divisors have the same degree (or in other words only one elementary divisor occurs with nonzero multiplicity). Then by repeated use of Lemma 2 we can achieve a decomposition of all of V into finite-dimensional subspaces.

Actually this assumption of homogeneity is unnecessarily severe. For suppose the elementary divisor of highest degree occurs with only finite multiplicity. Then by a finite number of applications of Lemma 2(c) we account completely for this highest elementary divisor, and we are able to proceed. Moreover, a decomposition of all of V can be obtained in this way provided T is "almost homogeneous" in the sense that the only elementary divisor of infinite multiplicity is the one of lowest degree. In summary, we have proved the following theorem.

THEOREM 1. *Let F be any field and V a vector space of countable dimension over F with a nonsingular symmetric inner product; let T be a nilpotent self-adjoint linear transformation on V , with the property that the only elementary divisor of infinite multiplicity is the one of lowest degree. Then V is an orthogonal direct sum of finite-dimensional subspaces invariant under T .*

To get a more precise result than Theorem 1 we add the assumption that F is algebraically closed⁵ and of characteristic not 2. Then finite-dimensional theory (§2) shows that the summands in Theorem 1 are determined up to orthogonal similarity by their elementary divisors. We may state:

THEOREM 2. *Assume, in addition to the hypothesis of Theorem 1, that F is algebraically closed and of characteristic not 2. Then T is determined up to orthogonal similarity by its elementary divisors. In other words, two such linear transformations are orthogonally similar if and only if they are similar.*

REMARK. Theorems 1 and 2 may fail if the hypothesis of countability is dropped, as easily constructed examples show. Moreover, the hypothesis of almost homogeneity is likewise essential, as we shall see in §5.

5. The case $T^2=0$. We now leave the almost homogeneous case treated in Theorems 1 and 2. The complications appear to become rather severe, and we shall not attempt to go beyond the case $T^2=0$.

Let then T be a self-adjoint linear transformation on a vector space with a nonsingular symmetric inner product, and suppose $T^2=0$. We proceed to collect the obvious invariants. Let R be the range of T , N the null space. We have that $N=R'$, the orthogonal complement of R (this is true for any self-adjoint T); also $R \subset N$ (as a consequence of $T^2=0$). By taking orthogonal complements we find the inclusions and equalities:

$$V \supset N = R' = N'' \supset N' = R'' \supset R \supset 0.$$

This gives rise to four quotient spaces: V/N , N/N' , N'/R , and R . However R and V/N have the same dimension; so we are left with three invariants, the dimensions of N/N' , N'/R , and R . We shall call these dimensions the *cardinal number invariants* of T . It is worth noting that $\dim R$ is the multiplicity of the elementary divisor λ^2 , while $\dim N/R$ is that of the elementary divisor λ ; so what we have done is to refine the latter invariant by splitting it into two parts.

We shall now prove that in the countable case there are no further orthogonal invariants.

THEOREM 3. *Let F be a field of characteristic not 2 in which every element is a square; let V be a vector space of countable dimension over F*

⁵ Actually we need only assume that every element in F is a square. But this refinement is of minor interest, since the reduction of §3 requires algebraic closure.

with a nonsingular symmetric inner product; let T be a self-adjoint linear transformation on V satisfying $T^2=0$. Then T is determined up to orthogonal similarity by its three cardinal number invariants.

It will be convenient first to prove several lemmas.

LEMMA 3. Suppose, in addition to the hypothesis of Theorem 3, that $\dim R$ is infinite, and let G be a finite-dimensional subspace of V , invariant under T . Then: (a) for any α in F , there exists v in G' with $(vT, v)=1$, $(v, v)=\alpha$; (b) there exists v in G' but not in $G+N$, satisfying $(v, v)=(vT, v)=0$.

PROOF. (a) We begin by noting that it cannot be the case that $(aT, b)=0$ for all a, b in G' , for then $G'T$ would annihilate G' , which means $G'T \subset G''=G$, $G' \subset G+N$, contradicting the hypothesis that $\dim V/N=\dim R$ is infinite. From this it next follows (since the characteristic is not 2) that (cT, c) cannot be 0 for all c in G' . By multiplying c by a suitable scalar, we can arrange $(cT, c)=1$. We then take $v=c-\lambda cT$, where $2\lambda=(c, c)-\alpha$.

(b) Using part (a), we choose r in G' with $(rT, r)=1$, $(r, r)=0$. Then by another use of (a), with G replaced by the subspace spanned by G , r , and rT , we choose s orthogonal to G , r , and rT so as to satisfy $(sT, s)=1$, $(s, s)=0$. We then take $v=r+is$, where $i^2=-1$. We have $(vT, v)=(v, v)=0$; moreover v cannot be in $G+N$, for it is not orthogonal to rT , and the latter lies in $G' \cap R$ which annihilates $G+N$.

LEMMA 4. The hypothesis is the same as in Lemma 3. Suppose $w \in V$ and $\alpha, \beta \in F$ are given. Then there exists an element v in G' such that $y=w+v$ is not in $G+N$, and $(y, y)=\alpha$, $(yT, y)=\beta$.

PROOF. We apply Lemma 3(a) and get r orthogonal to G , w , and wT with $(r, r)=(rT, r)=1$. Then we apply Lemma 3(a) again and get s orthogonal to G , w , wT , r , and rT , with $(s, s)=0$, $(sT, s)=1$. The equations

$$\begin{aligned}(w + \lambda r + \mu s, w + \lambda r + \mu s) &= \alpha, \\ ((w + \lambda r + \mu s)T, w + \lambda r + \mu s) &= \beta\end{aligned}$$

can be solved for λ and μ in F . Then the choice $v=\lambda r+\mu s$ is suitable except for the possibility that $z=w+\lambda r+\mu s$ is in $G+N$. If indeed z is in $G+N$, we make a further adjustment. By Lemma 3(b) we find p orthogonal to G , z , and zT , not in $G+N$, and satisfying $(p, p)=(pT, p)=0$. Then substitution of $z+p$ for z does not upset the values of (z, z) and (zT, z) , and $z+p$ does not lie in $G+N$. Hence $v=\lambda r+\mu s+p$ satisfies all the requirements.

LEMMA 5. *Suppose in addition to the hypothesis of Theorem 3 that $\dim N/N'$ is infinite, and let G be a finite-dimensional subspace of V . Let $w \in N$ and $\alpha \in F$ be given. Then there exists an element v in $N \cap G'$ such that $y = w + v$ is not in $G + N'$, and $(y, y) = \alpha$.*

The proof of Lemma 5 is very much like (and easier than) the proof of Lemma 4, and we accordingly leave it to the reader.

The final lemma is an elementary one and is valid under very general circumstances.

LEMMA 6. *Let V be a vector space with a symmetric inner product, and let G and K be subspaces of V , with G finite-dimensional. Let $z_1, \dots, z_r$ be a basis of a complement of $G \cap K'$ in G , and $\alpha_1, \dots, \alpha_r$ any scalars. Then there exists an element w in K with $(w, z_i) = \alpha_i$ for all i .*

PROOF. Each element of K induces a functional on G which vanishes on $G \cap K'$, and thus induces a functional on $G/(G \cap K')$. In this way, K appears as a total linear space of functionals on the finite-dimensional vector space $G/(G \cap K')$. Consequently K induces all functionals on $G/(G \cap K')$, and this is precisely what the lemma asserts.

PROOF OF THEOREM 3. We begin the proof by making two normalizations. The case where $\dim R$ is finite falls precisely under Theorem 2 and need not be considered here; we shall therefore assume $\dim R$ infinite. Next we take up the possibility that $\dim N/N'$ is finite. If so, let Y be a complement of N' in N . Then Y is nonsingular; for if $y \in Y$ annihilates all of Y , then y annihilates $Y + N' = N$, $y \in N'$, a contradiction. Moreover Y is invariant under T since in fact $YT = 0$. We have the decomposition $V = Y \oplus Y'$, and we can confine ourselves to the summand Y' , on which $N' = N$. In summary: we may assume that either $N' = N$ or that $\dim N/N'$ is infinite.

In carrying out the proof of Theorem 3 we shall not directly set up a canonical form for T , for this would necessarily be somewhat complicated. Instead, following ideas initiated by Mackey in [3], we suppose that a second linear transformation S is given, again self-adjoint and with $S^2 = 0$, and with the same cardinal number invariants as T . We write Q for the range of S , M for its null space. Our task is to find an orthogonal linear transformation U such that $U^{-1}SU = T$. We shall build U stepwise. Suppose we have reached finite-dimensional subspaces F and G and a one-to-one linear transformation U of F onto G satisfying

(A) F, G are invariant under S, T respectively,

(B) If $u \in G \cap R$, then there exists v in G with $vT = u$, and a similar assumption on F ,

- (C) $U^{-1}SU = T$ holds on F ,
 (D) U preserves the inner product, •
 (E) U maps $F \cap Q, F \cap M', F \cap M$ onto $G \cap R, G \cap N', G \cap N$ respectively.

We now describe the procedure for extending the map U one step further. It is understood that this is to be done by adjunctions alternating between F and G , and so we make sure that U is ultimately defined on all of V and has all of V for its range. For definiteness, we suppose that F is being enlarged, and we have to find a corresponding enlargement of G .

Before launching the discussion, we pause to make yet another normalization. This time we are concerned about the possibility that $\dim M'/Q = \dim N'/R$ is finite. If so, let X and Y be complements of Q, R in M', N' respectively. As the very first step of the construction, we pick U to be any one-to-one map of X onto Y . Since X and Y are annihilated by S and T , and the inner product vanishes identically on X and Y , it is clear that properties (A) to (E) above are satisfied. If we agree to start the construction this way, we shall be entitled to assume the following: either $\dim N'/R$ is infinite, or else $F+Q$ contains M' and $G+R$ contains N' .

Now choose once for all a fixed basis of V . Let x be the first basis element not in F . We must distinguish four cases.

- I. x is in $F+Q$,
- II. x is in $F+M'$, but not in $F+Q$,
- III. x is in $F+M$, but not in $F+M'$,
- IV. x is not in $F+M$.

Case I. We can suppose that x is actually in Q . Then there exists x_1 with $x_1S=x$. The element x_1 cannot be in $F+M$, for then $x_1S=x$ is in F . This switches us to Case IV.

Case II. We can suppose x is in M' . Note that by the final normalization made above, $\dim N'/R$ is infinite. We are seeking a suitable choice for an element y to play the role of xU , so as to continue to satisfy conditions (A)–(E). Since $(x, x)=0$, $xS=0$, and the same will be true of any y in N' , only (D) and (E) cause us any concern; to satisfy them, y must be in N' but not in $G+R$, and

$$(3) \quad (x, a) = (y, aU) \quad \text{for every } a \text{ in } F.$$

Let $t_1, \dots, t_r$ be a basis for a complement of $F \cap M$ in F . By (E), U maps $F \cap M$ onto $G \cap N$. Hence $t_1U, \dots, t_rU$ is a basis for a complement of $G \cap N$ in G . We now quote Lemma 6, with N' playing the role of K (note that $N''=N$), and $\alpha_i=(x, t_i)$; we deduce the existence of w in N' satisfying $(x, t_i)=(w, t_iU)$. Since $(x, M)=(w, N)$

$=0$, it follows that the choice $y=w$ satisfies (3). But there remains the possibility that w is in $G+R$. If this is the case we adjust w by setting $y=w+v$ with $v \in G' \cap N'$; this does not disturb equation (3). If we are unable to escape from $G+R$ in this way, then $G' \cap N'$ must be contained in $G+R$; but this contradicts the infinite-dimensionality of N'/R .

Case III. We may assume $x \in M$. By a normalization made above, $\dim N/N'$ is infinite. The requirements on $y=xU$ now are: y must be in N but not in $G+N'$, $(y, y)=(x, x)$, and y satisfies (3). In view of the fact that U maps $F \cap M'$ onto $G \cap N'$, Lemma 6 is applicable (just as in Case II) to yield an element w in N which satisfies (3). Then Lemma 5 passes us to an element y satisfying all requirements.

Case IV. From the fact that x is not in $F+M$, it follows that xS is not in F . For suppose $xS=a \in F$. By (B) there exists b in F with $bS=a$. Then $(x-b)S=0$, $x-b \in M$, $x \in F+M$, a contradiction. It is easy to verify further the following statements: x, xS are linearly independent mod F , and if we denote by F_1 the space spanned by F, x , and xS , then $F_1 \cap Q, F_1 \cap M', F_1 \cap M$ are obtained from $F \cap Q, F \cap M', F \cap M$ by adjunction of xS .

Similar remarks will apply to any element y , so long as it is selected outside of $G+N$. So we see that (A)–(E) will be fulfilled if we pick $y=xU$ outside of $G+N$, satisfying (3) and also satisfying $(y, y)=(x, x)$, $(yT, y)=(xT, x)$. To do this we first pick any w satisfying (3). Then by Lemma 4 we pass to an element y meeting all requirements. This concludes the proof of Theorem 3.

Thus we have settled the uniqueness, under orthogonal similarity, of a linear transformation with given cardinal number invariants. It is a further fact that a T always exists with prescribed invariants. Here is an example with $\dim R = \infty$, $\dim N'/R = 1$, $\dim N/N' = 0$. By taking a suitable direct sum of copies of this example, as well as trivial examples, one can look after an arbitrary combination of invariants.

EXAMPLE. V has a basis consisting of $z, x_1, x_2, \dots$, and $y_1, y_2, \dots$. The inner product is defined by $(z, z)=(z, y_i)=0$, $(z, x_i)=1$, $(x_i, x_i)=(y_i, y_i)=0$, $(x_i, y_i)=1$, $(x_i, x_j)=(x_i, y_j)=(y_i, y_j)=0$ for $i \neq j$. We leave to the reader the verification of the nonsingularity of this inner product, but remark that infinite-dimensionality is really needed. T is defined by $zT=y_1T=0$, $x_iT=y_i$. Then R is spanned by the y 's while $N=N'$ =the space spanned by the y 's and z .

This example makes it plain that the elementary divisors are no longer a complete set of invariants in the infinite-dimensional case, or in other words, similarity does not imply orthogonal similarity.

However it is easy to verify that the three cardinal number invariants are preserved under similarity by a *continuous* linear transformation. So it remains possible that under very general hypotheses continuous similarity implies orthogonal similarity. To be sure, this would merely reduce one unsolved problem to another.

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