absolutely convergent. Hence

$$B\{\phi_n(r), x\} = B\{s_n, rx\}.$$

It follows that $(B) \lim s_n = s$ implies $(B) \lim \phi_n(r) = s$.

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A NOTE ON INDEFINITE INTEGRALS

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I. Throughout the paper $f(x)$ will denote a given function, real-valued and Lebesgue integrable on the interval $X \equiv (0 \leq x \leq 1)$. We denote by E the generic measurable subset of X and introduce the following definitions.

$$(1.1) \quad F(E) = F(E; f) = \int_E f(x) dx.$$

(1.2) $B_*(\alpha) \equiv B_*(\alpha; f)$ and $B^*(\alpha) \equiv B^*(\alpha; f)$ are, respectively, the greatest lower and least upper bounds of $F(E)$ taken over all sets E of measure $|E| = \alpha$ ($0 \leq \alpha \leq 1$).

Regarded as a function of E , $F(E)$ is called a generalized indefinite integral of $f(x)$ [4] or simply an indefinite integral of $f(x)$ [1]. In this section we develop the principal properties of the functions $B_*(\alpha)$ and $B^*(\alpha)$, and then obtain as a main result (see (1.9)) the fact that the values of $F(E)$ for $|E| = \alpha$ comprise the closed interval from $B_*(\alpha)$ to $B^*(\alpha)$. This is an extension of the known fact that $F(E)$ assumes all values between its optimum bounds, where no restrictions are placed on the measures of the sets involved [4]. In the second section the results of the first are applied to the problem of defining a mean value for $F(E)$ as E ranges over the measurable subsets of X .

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(1.3) The functions $B_*(\alpha)$ and $B^*(\alpha)$ are bounded on the interval $0 \leq \alpha \leq 1$; and $B_*(0) = B^*(0) = 0$, $B_*(1) = B^*(1) = F(X)$.

These facts follow directly from (1.1), (1.2), and the inequality $|F(E)| \leq F(X; |f|)$ for all E .

(1.4) $B^*(\alpha) = F(X) - B_*(1 - \alpha)$ for $0 \leq \alpha \leq 1$.

If $|E| = \alpha$, then $|X - E| = 1 - \alpha$ and $F(E) = F(X) - F(X - E)$. The relation stated is an immediate consequence.

(1.5) If $f(x)$ is nondecreasing, then $B_*(\alpha) = \int_0^\alpha f(x) dx$ and $B^*(\alpha) = \int_{1-\alpha}^1 f(x) dx$ for $0 \leq \alpha \leq 1$.

Let E be an arbitrary set with $0 < |E| = \alpha < 1$ and let $A \equiv (0 \leq x \leq \alpha)$. Define E_1 as EA , E_2 as $E - E_1$, and E_3 as $A - E_1$, so that $|E_2| = |E_3|$. On E_2 , $f(x) \geq f(\alpha)$; and on E_3 , $f(x) \leq f(\alpha)$. Consequently, $F(E) = F(E_1) + F(E_2) = F(A) + F(E_2) - F(E_3) \geq F(A)$. This establishes the first relation. The second now follows from (1.4).

It is convenient at this point to introduce a certain rearrangement of the function $f(x)$ defined in the following manner. Let n be a given positive integer and set $N = 2^n$. Divide the interval $0 \leq x < 1$ into consecutive subintervals $J_i^n \equiv \{(i-1)/N \leq x < i/N\}$ ($i = 1, 2, \dots, N$), and let S_n denote the collection of intervals $(J_1^n, J_2^n, \dots, J_N^n)$. Let $Q \equiv (J_{k_1}^n, J_{k_2}^n, \dots, J_{k_m}^n)$, $m \leq N$, be a specified subcollection of S_n composed of distinct intervals, with the enumeration so chosen that $F(J_{k_1}^n) \leq F(J_{k_2}^n) \leq \dots \leq F(J_{k_m}^n)$. Let $(J_{k_{m+1}}^n, \dots, J_{k_N}^n)$ denote the remaining intervals of S_n . We set $h_i = (k_i - 1)/N - (i - 1)/N$ and define a function $f_n(x; Q)$ as $f(x + h_i)$ for x in J_i^n ($i = 1, 2, \dots, N$). This rearranged function, $f_n(x; Q)$, will be useful in the sequel, in particular when Q is S_n itself, in which case $\int_0^1 f_n(x; S_n) dx = F(X)$.

Since the next theorem involves the notion of a convex function we state here the following definition. A function $g(\alpha)$ is said to be convex on the interval $0 \leq \alpha \leq 1$ if for every pair of points P_1, P_2 on the graph of $g(\alpha)$, the points of the arc P_1P_2 are below, or on, the chord P_1P_2 [7]. Such functions are easily seen to be continuous. In case $-g(\alpha)$ is convex we shall say that $g(\alpha)$ is concave.

(1.6) $B_*(\alpha)$ is a convex function on the interval $0 \leq \alpha \leq 1$.

Suppose first that $f(x)$ is nondecreasing. Then $B_*(\alpha) = \int_0^\alpha f(x) dx$ by (1.5), and the convexity follows from a known theorem [7]. In the general case we can determine for each $n = 1, 2, 3, \dots$ an integer i_n and a step function $\phi_n(x)$, constant on each interval of S_{i_n} , such that $\int_0^1 |f(x) - \phi_n(x)| dx \equiv \epsilon_n \rightarrow 0$. Let $\phi_{n, i_n}(x; S_{i_n}) \equiv \phi_n^*(x)$ denote the rearrangement of $\phi_n(x)$ as described above. For each n the sets of real numbers $F(E; \phi_n)$ and $F(E; \phi_n^*)$, obtained as E varies over all sets of measure α , are evidently identical. Therefore $B_*(\alpha; \phi_n) = B_*(\alpha; \phi_n^*)$, and the convexity of $B_*(\alpha; \phi_n)$ follows from the fact that $\phi_n^*(x)$ is

a nondecreasing function of x . Furthermore, we have $|F(E; f) - F(E; \phi_n)| \leq \epsilon_n$ for all E and all n , and this implies $|B_*(\alpha; f) - B_*(\alpha; \phi_n)| \leq \epsilon_n$, where $\epsilon_n \rightarrow 0$. Thus $B_*(\alpha; f)$, as the limit of a sequence of convex functions, is itself convex. This completes the proof.

(1.7) $B^*(\alpha)$ is a concave function on the interval $0 \leq \alpha \leq 1$.

This follows from (1.4) and (1.6).

(1.8) If $f(x)$ is not constant almost everywhere, then $B_*(\alpha) < \alpha F(X) < B^*(\alpha)$ for $0 < \alpha < 1$; and conversely.

In view of (1.4) it suffices to suppose that $B_*(\alpha_0) = \alpha_0 F(X)$ for some α_0 ($0 < \alpha_0 < 1$). Then by (1.3) the graph of the convex function $B_*(\alpha)$ has three points in common with a chord, and therefore coincides with that chord [2]. Consequently we have $B_*(\alpha) = \alpha F(X)$ for $0 \leq \alpha \leq 1$. From (1.4) we conclude then that $B_*(\alpha) = \alpha F(X) = B^*(\alpha)$ for $0 \leq \alpha \leq 1$. This implies in particular that $\int_0^x f(t) dt = x F(X)$ for all x in X . On differentiating each side we find that $f(x) = F(X)$ almost everywhere—a contradiction. The truth of the converse statement is evident.

(1.9) For each α ($0 \leq \alpha \leq 1$) and each β ($B_*(\alpha) \leq \beta \leq B^*(\alpha)$) there exists a set E of measure α such that $F(E) = \beta$.

The proof depends upon the following relations wherein $f_n(x; S_n)$ is the rearrangement of $f(x)$ described above.

$$(1.10) \quad \lim_n \int_0^\alpha f_n(x; S_n) dx = B_*(\alpha; f) \quad (0 \leq \alpha \leq 1).$$

$$(1.11) \quad \lim_n \int_{1-\alpha}^1 f_n(x; S_n) dx = B^*(\alpha; f)$$

We may assume $0 < \alpha < 1$ and we first confine attention to α of the form $\rho = \kappa/2^\mu$. Then for all $n \geq \mu$ we have $\int_0^\rho f_n(x; S_n) dx \leq \int_0^\rho f_{n-1}(x; S_{n-1}) dx$. For in the notations introduced above the first integral can be written as the sum $\sum_{\nu=1}^p F(J_{k_\nu}^n)$, where $p = \kappa 2^{n-\mu}$, and the second can be expressed in the same form, $\sum_{\nu=1}^p F(J_{l_\nu}^n)$, with, in general, a different arrangement of the J intervals. The inequality as stated now follows from the fact that a partial sum of a nondecreasing arrangement of a set of real numbers never exceeds the corresponding partial sum of any other arrangement. Since $\int_0^\rho f_n(x; S_n) dx \geq B_*(\rho; f)$ for all n , we conclude that $\lim_n \int_0^\rho f_n(x; S_n) dx$ exists, and equals, say, $\lambda(\rho) \geq B_*(\rho; f)$. Assuming that the inequality sign holds we arrive at a contradiction as follows. Let $\epsilon > 0$ be chosen such that $B_*(\rho; f) + \epsilon < \lambda(\rho)$, and then fix $\delta > 0$ so that $|F(e)| < \epsilon/4$ for all sets e with $|e| < \delta$. Determine next a set E with $|E| = \rho$ such that $F(E) < B_*(\rho; f)$

$+\epsilon/4$. Corresponding to the preceding δ it is readily seen that there exist an integer $n \geq \mu$ and a set of intervals $Q = (J_{k_1}^n, J_{k_2}^n, \dots, J_{k_m}^n)$, contained in S_n , such that $|R - E| < \delta$, $|E - R| < \delta$, $|r - \rho| < \delta$, where R is the set-theoretic sum of the intervals composing Q and $r = |R|$. We now have

$$\begin{aligned} \int_0^\rho f_n(x; S_n) dx &\leq \int_0^\rho f_n(x; Q) dx = \int_0^r f_n(x; Q) dx + \int_r^\rho f_n(x; Q) dx \\ &= F(R) + \int_r^\rho f_n(x; Q) dx = F(E) + F(R - E) \\ &\quad - F(E - R) + \int_r^\rho f_n(x; Q) dx \\ &< B_*(\rho; f) + \epsilon < \lambda(\rho). \end{aligned}$$

This contradiction establishes (1.10) for α of the form ρ . Now let α be given arbitrarily and let $\{\rho_k\}$ be a sequence of numbers of the form ρ converging to α . We then have

$$\begin{aligned} B_*(\alpha; f) &= \lim_k B_*(\rho_k; f) = \lim_k \lim_n \int_0^{\rho_k} f_n(x; S_n) dx \\ &= \lim_n \lim_k \int_0^{\rho_k} f_n(x; S_n) dx = \lim_n \int_0^\alpha f_n(x; S_n) dx, \end{aligned}$$

wherein we use the continuity of $B_*(\alpha; f)$ implied by (1.6), and the fact that the limits can be interchanged since the limit over k is uniform with respect to n by the absolute continuity of the integral. This completes the proof of (1.10). The relation (1.11) now follows from (1.4), (1.10), and the fact that $\int_0^1 f_n(x; S_n) dx = F(X)$.

PROOF OF (1.9). *Case 1:* $B_*(\alpha) < \beta < B_*(\alpha)$. Using (1.10) and (1.11) we fix n so large that

$$\int_0^\alpha f_n(x; S_n) dx < \beta < \int_{1-\alpha}^1 f_n(x; S_n) dx.$$

If we set $\psi(\lambda) = \int_\lambda^{\lambda+\alpha} f_n(x; S_n) dx$ for $0 \leq \lambda \leq 1 - \alpha$, then $\psi(0) < \beta$, $\psi(1 - \alpha) > \beta$, and consequently there exists a λ_0 ($0 < \lambda_0 < 1 - \alpha$) for which $\psi(\lambda_0) = \beta$. It is clear that the set $\lambda_0 \leq x \leq \lambda_0 + \alpha$, by the inverse of the rearrangement leading to $f_n(x; S_n)$, corresponds to a set E of measure α such that $F(E) = \beta$. It is furthermore apparent that E is merely a finite set of intervals. Consequently, by adjoining or deleting the proper end points, the critical set E of the theorem may be

chosen either perfect or open.

Case 2: (a) $\beta = B_*(\alpha)$; (b) $\beta = B^*(\alpha)$. It will suffice to give the proof in subcase (b) since the result in subcase (a) will then follow from (1.4). For this purpose we introduce the function $\mu(c) \equiv |M\{f(x) \geq c\}|$ for $-\infty < c < +\infty$, where $M\{\dots\}$ is the set of all x in X satisfying the condition $\dots$ in brackets. The following properties of $\mu(c)$ are easily verified: it is nonincreasing; $\mu(-\infty) = 1$, $\mu(+\infty) = 0$; $\mu(c-0) = \mu(c)$; and $\mu(c) - \mu(c+0) = |M\{f(x) = c\}|$. We need also the fact that

$$(1.12) \quad B^*(\alpha; f - c) = B^*(\alpha; f) - c\alpha.$$

Now if α is given, the preceding properties imply the existence of a unique c such that $\mu(c+0) \leq \alpha \leq \mu(c)$. Then α is of the form $\mu(c+0) + \theta|M\{f(x) = c\}|$, where θ is a certain constant ($0 \leq \theta \leq 1$). Let E_1 be any subset of $M\{f(x) = c\}$ of measure $\theta|M\{f(x) = c\}|$, and set $E = M\{f(x) > c\} + E_1$. The set E is of measure α ; and on E , $f(x) - c \geq 0$, while on $X - E$, $f(x) - c \leq 0$. Consequently, $B^*(\alpha; f - c) = F(E; f - c) = F(E; f) - c\alpha$, and $F(E; f) = B^*(\alpha; f)$ follows from (1.12). This completes the proof of (1.9). We add one remark. If $f^*(x)$ is the characteristic function of an arbitrary set E^* of measure α , then $B^*(\alpha; f^*) = F(E^*)$. Hence, in general, it will be impossible to choose a critical set in Case 2 of the simple structure possible for its counterpart in Case 1.

From (1.3), (1.6), and (1.7) it follows that the graphs of $B_*(\alpha)$ and $B^*(\alpha)$ bound a closed convex domain which we denote by $D = D(f)$. In view of the symmetry implied by (1.4) it is obvious that D has the point $(1/2, F(X)/2)$ as its geometrical center. Hence, by (1.8), if $f(x)$ is not constant almost everywhere, the familiar elementary formula,

$$(1.13) \quad \frac{1}{2} \int_0^1 [B^{*2}(\alpha) - B_*^2(\alpha)] d\alpha \bigg/ \int_0^1 [B^*(\alpha) - B_*(\alpha)] d\alpha = \frac{1}{2} F(X),$$

is valid. Another interpretation of (1.13) is given in the next section. A second formula to be used later is the following.

$$(1.14) \quad \int_0^1 [B_*(\alpha) + B^*(\alpha)] d\alpha = F(X).$$

This follows directly from (1.4).

(1.15) *If $f(x)$ is nondecreasing, then the area of D , namely $\int_0^1 [B^*(\alpha) - B_*(\alpha)] d\alpha$, is equal to $\int_0^1 (2x - 1)f(x) dx$.*

To prove this we first observe that (1.5) gives

$$\int_0^1 \left(\int_{1-\alpha}^1 f(x) dx \right) d\alpha - \int_0^1 \left(\int_0^\alpha f(x) dx \right) d\alpha,$$

for the area in question. Since the function $g(x, \alpha) \equiv f(x)$ is integrable over the unit square in the (x, α) -plane, we may use the Fubini theorem to interchange the order of integration. The formula stated follows at once. If $f(x)$ is not nondecreasing the result is not in general valid, although there is a similar expression for the case of non-increasing $f(x)$.

For the sake of completeness it is of interest to ask whether the function $B_*(\alpha; f)$, which is uniquely determined by f , has any special properties beyond those implied by its convexity. The answer is in the negative. For let $b(\alpha)$ be a given function, convex on the interval $0 \leq \alpha \leq 1$, with $b(0) = 0$ and $b(1) = a$. Then there exists [7] a non-decreasing and integrable function $g(x)$ such that $b(\alpha) = \int_0^\alpha g(x) dx$ ($0 \leq \alpha \leq 1$). Therefore $B_*(\alpha; g) = b(\alpha)$ by (1.5). On the other hand, there will exist in general functions g^* , distinct from g , for which $B_*(\alpha; g^*) = b(\alpha)$; for example, the rearranged functions $g_n(x; S_n)$.

II. The problem of assigning a mean value to the set of sums of all subseries of a given absolutely convergent series $\sum u_n = s$ has been discussed by the writer, using the following approach [3]. Let $t = .a_1 a_2 a_3 \cdots$ be the nonterminating dyadic expansion of t and let $\phi(t) \equiv \sum a_n u_n$ ($0 < t \leq 1$). Then the mean value in question was defined and evaluated by the formula

$$(2.1) \quad \int_0^1 \phi(t) dt = \frac{s}{2}.$$

It was shown later by Pollard [5] that (2.1) is valid if and only if $\sum u_n$ and $\sum u_n^2$ both converge.

If we replace $\sum u_n$ by the Lebesgue integral $F(X)$, the problem arises of defining a mean value, $m[F(E)]$, for the set of values of all the "subintegrals" $F(E)$, obtained as E varies over the measurable subsets of X . This problem is more complicated than the preceding since there is no simple independent variable available that will take into account the frequency (namely, 2^c) with which $F(E)$ assumes an admissible value β . However, we may frame a definition on the basis of $\alpha = |E|$ as the independent variable if we agree to assign to each value $\beta = F(E)$, assumed on a set of measure α , the arbitrary frequency 1. This is equivalent to identifying E_1 and E_2 of the same measure α if $F(E_1) = F(E_2)$. With this convention in mind the results

of §I lead to two possible definitions of $m[F(E)]$, each consistent with the analogous result (2.1).

Let M_α be the set of all β such that $B_*(\alpha) \leq \beta \leq B^*(\alpha)$ for $0 \leq \alpha \leq 1$. Each such β , by (1.9), is a value of $F(E)$ for $|E| = \alpha$. For the average value of $F(E)$ over the set M_α we take $A_\alpha \equiv [B_*(\alpha) + B^*(\alpha)]/2$. We now employ a weighted integral mean of the form $\int_0^1 w(\alpha) A_\alpha d\alpha / \int_0^1 w(\alpha) d\alpha$. In the first definition, $m_1[F(E)]$, we choose $w(\alpha) \equiv 1$, and from (1.14) we obtain $m_1[F(E)] = F(X)/2$. In the second definition, $m_2[F(E)]$, we assume that $f(x)$ is not constant almost everywhere, and set $w(\alpha) = |M_\alpha|$. Then $w(\alpha) > 0$ for $0 < \alpha < 1$ by (1.8), and from (1.13) we find that $m_2[F(E)] = F(X)/2 = m_1[F(E)]$.

It may be remarked finally that there remains the possibility of discussing the range of $F(E)$ when E is regarded as an element of a metric space in which the distance between the sets E_1 and E_2 is defined as the measure $|E_1 + E_2 - E_1 E_2|$ of the "symmetric difference" of E_1 and E_2 (see [6] and the references listed there). Such an approach might lead to alternative proofs of certain of the results given here (for example, (1.9)), and perhaps to a definition of $m[F(E)]$ that takes into account the frequency of $F(E) = \beta$.

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