ON PRODUCTS OF SUMMABILITY METHODS

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1. At the recent International Congress I. M. Sheffer asked me the following question: Given a sequence $\{s_n\}$; form the sequence $\{\sigma_n^\alpha\}$ of the Cesàro means of order α ($\alpha > 0$) corresponding to $\{s_n\}$. If $\{s_n\}$ is summable Abel, is it true that $\{\sigma_n^\alpha\}$ is also summable Abel?

A more general question is: Suppose that A and B are two regular summability methods for sequences $\{s_n\}$. Denote by AB the iteration product which associates with a given sequence the A transform of its B transform; when does A summability imply AB summability?

We shall show in §2 that the answer is affirmative when B is (C, α) and A is Abel summability.[1] In §3 we generalize this result to Laplace transforms and Riesz summability. In §4 we discuss the iteration product of Cesàro and Borel summability, and also Euler and Borel summability.

2. Let $f(x) = \sum_0^\infty a_n x^n = (1-x)\sum s_n x^n$. Abel summability of $\sum a_n$ to s is

$$A \lim s_n = \lim_{x\to 1} (1 - x) \sum s_n x^n = s.$$

We define s_n^α by the identity

$$f(x) = (1 - x)^{\alpha+1} \sum_0^\infty s_n^\alpha x^n,$$

and γ_n^α by

$$(1 - x)^{-\alpha-1} = \sum_0^\infty \gamma_n^\alpha x^n,$$

so that

$$\gamma_n^\alpha = C_{n+\alpha,n} = \frac{\Gamma(\alpha + n + 1)}{n!\Gamma(\alpha + 1)}. \tag{2.1}$$

We now have

$$\sigma_n^\alpha = \frac{s_n^\alpha}{\gamma_n^\alpha}, \qquad s_n^\alpha = \sum_{v=0}^n s_v \gamma_{n-v}^{\alpha-1}. \tag{2.2}$$

Presented to the Society, April 27, 1951; received by the editors February 7, 1951.

[1] For the case $\alpha = 1$ see [4, p. 189]; [3, p. 258]. Numbers in brackets refer to the literature at the end of this paper.

Furthermore

$$\frac{1}{\gamma_n^\alpha} = \alpha \int_0^1 \rho^n (1-\rho)^{\alpha-1} d\rho, \tag{2.3}$$

hence

$$\sum_0^\infty \sigma_n^\alpha x^n = \alpha \int_0^1 (1-\rho)^{\alpha-1} \sum s_n^\alpha (\rho x)^n d\rho$$
$$= \alpha \int_0^1 (1-\rho)^{\alpha-1} (1-\rho x)^{-\alpha-1} f(\rho x) d\rho.^2$$

The transformation

$$\rho x = 1 - t^{-1}, \qquad x d\rho = t^{-2} dt$$

yields

$$\sum_0^\infty \sigma_n^\alpha x^n = \frac{\alpha}{x} \int_1^{1/(1-x)} \left(t - \frac{t-1}{x}\right)^{\alpha-1} f(1-t^{-1}) dt.$$

Putting $x = 1-(y+1)^{-1}$ $(y \to \infty)$ yields

$$\sum_0^\infty \sigma_n^\alpha \left(1 - \frac{1}{y+1}\right)^n$$
$$= \alpha \left(1 + \frac{1}{y}\right)^\alpha \int_1^{y+1} \left(1 - \frac{t-1}{y}\right)^{\alpha-1} f(1-t^{-1}) dt.$$

Now set $t = u+1$; then

$$\frac{1}{y+1} \sum_0^\infty \sigma_n^\alpha \left(1 - \frac{1}{y+1}\right)^n = \left(1 + \frac{1}{y}\right)^{\alpha-1} \cdot \frac{\alpha}{y} \int_0^y \left(1 - \frac{u}{y}\right)^{\alpha-1} \phi(u) du, \tag{2.4}$$

where $\phi(u) = f(1-(u+1)^{-1})$. Now

$$\frac{\alpha}{y} \int_0^y \left(1 - \frac{u}{y}\right)^{\alpha-1} \phi(u) du$$

is the integral (C, α) transform of the function $\phi(u)$; denoting it by $(\overline{C}, \alpha)$ we have from (2.4)

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[2] The same formula appears in [2, p. 200]. See also E. C. Titchmarsh, *The theory of functions*, Oxford, 1932, p. 242, example 8.

$$A(C, \alpha)\{s_n\} = \left(1 + \frac{1}{y}\right)^{\alpha-1} (\overline{C}, \alpha)\{\phi(u)\},$$

from which our assertion follows, as the right side tends to s for $y \to \infty$.

3. Abel summability has been generalized to Dirichlet series

$$D(t) = \sum_0^\infty a_n e^{-\lambda_n t}, \qquad \text{where } 0 \leqq \lambda_0 < \lambda_1 < \cdots, \lambda_n \to \infty.$$

If the Dirichlet series converges for $t > 0$, and if $D(t) \to s$ as $t \to 0$, then we write $D_\lambda \sum a_n = s$. The method is regular. To Abel summability corresponds the case $\lambda_n = n$, $e^{-t} = x$. Furthermore

$$D(t) = t \sum_0^\infty s_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-ut} du.$$

Introducing the stepfunction

$$s(u) = \begin{cases} s_n, & \text{for } \lambda_n \leqq u < \lambda_{n+1}, \ n = 0, 1, 2, \cdots, \\ 0, & \text{for } 0 \leqq u < \lambda_0, \text{ if } \lambda_0 > 0, \end{cases} \tag{3.1}$$

we get

$$D(t) = t \int_0^\infty s(u) e^{-ut} du. \tag{3.2}$$

If $s_n \to s$, then $s(u) \to s$, $u \to \infty$, and $\lim_{t \to 0} D(t) = s$ defines the generalized limit of $s(u)$.

In general (3.2) is a regular transform of the function $s(u)$, the Laplace transform for which we write $L\{s(u), t\}$. The $(\overline{C}, \alpha)$ means of $s(u)$ are

$$C_\alpha(x) \equiv \alpha x^{-\alpha} \int_0^x (x - u)^{\alpha-1} s(u) du, \qquad \alpha > 0.$$

In the case of (3.1) if $\lambda_n \leqq x < \lambda_{n+1}$, then $C_\alpha(x)$ reduces to

$$R_\alpha(x) \equiv \alpha x^{-\alpha} \left\{ \sum_0^{n-1} s_v \int_{\lambda_v}^{\lambda_{v+1}} (x - u)^{\alpha-1} du + s_n \int_{\lambda_n}^x (x - u)^{\alpha-1} du \right\}$$
$$= x^{-\alpha} \sum_0^n a_v (x - \lambda_v)^\alpha.$$

These are Riesz's typical means of order α (see [1, chap. 4]). If $R_\alpha(x) \to s$, then the series $\sum a_n$ is called summable to $s(\lambda, \alpha)$. It is known that [1, p. 39] if $\sum a_n$ is summable (λ, α), then it is summable

D_λ to the same value. Similarly if $C_\alpha(x) \to s$ $(x \to \infty)$, then $L\{s(u), t\} \to s$ $(t \to 0)$. We have [1, p. 39]

$$D(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty s_\alpha(u) e^{-ut} du,$$

where $s_\alpha(x) = \alpha \int_0^x (x-u)^{\alpha-1} s(u) du$. Now

$$C_\alpha(x) = x^{-\alpha} s_\alpha(x),$$

hence

$$D(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty u^\alpha C_\alpha(u) e^{-ut} du.$$

We have

$$\Gamma(\alpha) = u^\alpha \int_0^\infty \rho^{\alpha-1} e^{-u\rho} d\rho, \qquad \alpha > 0,$$

so that

$$\begin{aligned} L\{C_\alpha(u), t)\} &= t \int_0^\infty C_\alpha(u) e^{-ut} du \\ &= \frac{t}{\Gamma(\alpha)} \int_0^\infty u^\alpha C_\alpha(u) e^{-ut} \int_0^\infty \rho^{\alpha-1} e^{-u\rho} d\rho du \\ &= \frac{t}{\Gamma(\alpha)} \int_0^\infty \rho^{\alpha-1} \int_0^\infty s_\alpha(u) e^{-u(t+\rho)} du d\rho \\ &= \alpha t \int_0^\infty \rho^{\alpha-1} (t+\rho)^{-\alpha-1} D(t+\rho) d\rho. \end{aligned}$$

Suppose that $L\{s(u), t\} \to s$ as $t \to 0$. Now $\alpha t \int_0^\infty \rho^{\alpha-1}(t+\rho)^{-\alpha-1} d\rho = 1$, hence

$$L\{C_\alpha(x), t\} - s = \alpha t \int_0^\infty \rho^{\alpha-1}(t+\rho)^{-\alpha-1}\{D(t+\rho) - s\} d\rho.$$

But it is easily shown that this expression tends to 0. Thus the theorem: If

$$L\{s(u), t\} \to s \qquad \text{as } t \to 0,$$

then

$$L\{C_\alpha(x), t\} \to s \qquad \text{as } t \to 0.$$

4. We now consider the product of Borel and (C, α) summability. The Borel transform of a sequence $\{s_n\}$ is

$$B\{s_n, x\} = e^{-x}\sum_0^\infty \frac{s_n}{n!} x^n.$$

We say that $(B)\lim s_n = s$, if $\lim_{x\to\infty} B\{s_n, x\} = s$. The product $B(C, \alpha)$ is the transform

$$B\{\overset{\alpha}{\sigma}_n, x\} = e^{-x}\sum \overset{\alpha}{\sigma}_n \frac{x^n}{n!}\cdot$$

Employing (2.1) and (2.2) we have

$$\begin{aligned}\sum_{n=0}^\infty \overset{\alpha}{\sigma}_n \frac{x^n}{n!} &= \Gamma(\alpha+1)\sum_{n=0}^\infty \frac{x^n}{\Gamma(\alpha+n+1)}\sum_{v=0}^n s_v \gamma_{n-v}^{\alpha-1} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\sum_{n=0}^\infty \frac{x^n}{\Gamma(\alpha+n+1)}\sum_{v=0}^n s_v \frac{\Gamma(\alpha+n-v)}{(n-v)!} \\ &= \alpha\sum_{v=0}^\infty s_v \sum_{n=v}^\infty x^n \frac{\Gamma(\alpha+n-v)}{(n-v)!\Gamma(\alpha+n+1)}\,.\end{aligned}$$

The interchange of summation is legitimate if the double sum is absolutely convergent. From (2.3)

$$\frac{v!\Gamma(\alpha+n-v)}{\Gamma(\alpha+n+1)} = \int_0^1 \rho^{\alpha+n-v-1}(1-\rho)^v d\rho,$$

hence

$$\begin{aligned}\sum_{n=v}^\infty x^n \frac{\Gamma(\alpha+n-v)}{(n-v)!\Gamma(\alpha+n+1)} &= \frac{1}{v!}\int_0^1 \rho^{\alpha-1}(1-\rho)^v \sum_{n=v}^\infty \frac{x^n}{(n-v)!}\rho^{n-v}d\rho \\ &= \frac{x^v}{v!}\int_0^1 \rho^{\alpha-1}(1-\rho)^v e^{\rho x}d\rho.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{v=0}^\infty |s_v| \sum_{n=v}^\infty x^n \frac{\Gamma(\alpha+n-v)}{(n-v)!\Gamma(\alpha+n+1)} & \\ &= \sum_{v=0}^\infty |s_v| \frac{x^v}{v!}\int_0^1 \rho^{\alpha-1}(1-\rho)^v e^{\rho x}d\rho \\ &< \int_0^1 \rho^{\alpha-1}e^{\rho x}\left(\sum_{v=0}^\infty |s_v| \frac{x^v}{v!}\right)d\rho.\end{aligned}$$

By assumption the sum $\sum |s_n| x^n/n!$ is an entire function, hence the double sum on the left is convergent for all $x>0$; this proves that $\sum \sigma_n^\alpha x^n/n!$ is an entire function, and

$$\sum_{n=0}^{\infty} \sigma_n^\alpha \frac{x^n}{n!} = \alpha \sum_{v=0}^{\infty} s_v \frac{x^v}{v!} \int_0^1 \rho^{\alpha-1}(1-\rho)^v e^{\rho x} d\rho$$

$$= \alpha \int_0^1 \rho^{\alpha-1} e^{\rho x} \sum_0^{\infty} s_v \frac{x^v(1-\rho)^v}{v!} d\rho$$

$$= \alpha \int_0^1 \rho^{\alpha-1} e^{\rho x} e^{(1-\rho)x} B\{s_n, (1-\rho)x\} d\rho.$$

Finally

$$B\{\sigma_n^\alpha, x\} = \alpha \int_0^1 \rho^{\alpha-1} B\{s_n, (1-\rho)x\} d\rho$$

$$= \frac{\alpha}{x} \int_0^x \left(1 - \frac{t}{x}\right)^{\alpha-1} B\{s_n, t\} dt,$$

or

$$B(C, \alpha)\{s_n\} = (\overline{C}, \alpha) B\{s_n, t\}.$$

It follows that (B) lim $s_n = s$ implies (B) lim $\sigma_n^\alpha = s$. For $\alpha = 1$ this result (with a simple proof) was communicated to me by J. Barlaz at the summer meeting, 1949, in Boulder, Colorado.

We finally consider the product of Borel and the generalized Euler transform. The Euler transform E_r is defined by

$$\phi_n(r) = \sum_{v=0}^{n} C_{n,v} r^v (1-r)^{n-v} s_v.$$

It was shown by K. Knopp that this transform is regular if and only if r is real and $0 < r \leqq 1$. Now

$$B\{\phi_n(r), x\} = e^{-x} \sum_0^{\infty} \phi_n(r) \frac{x^n}{n!} = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{v=0}^{n} C_{n,v} r^v (1-r)^{n-v} s_v$$

$$= e^{-x} \sum_{v=0}^{\infty} s_v \frac{r^v}{v!} (1-r)^{-v} \sum_{n=v}^{\infty} \frac{x^n (1-r)^n}{(n-v)!}$$

$$= e^{-x} \sum_0^{\infty} s_v \frac{(rx)^v}{v!} e^{(1-r)x} = B\{s_n, rx\}.$$

The interchange of summation is legitimate, the double series being absolutely convergent. Hence

$$B\{\phi_n(r),\ x\} = B\{s_n,\ rx\}.$$

It follows that (B) lim $s_n = s$ implies (B) lim $\phi_n(r) = s$.

References

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A NOTE ON INDEFINITE INTEGRALS

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I. Throughout the paper $f(x)$ will denote a given function, real-valued and Lebesgue integrable on the interval $X \equiv (0 \leqq x \leqq 1)$. We denote by E the generic measurable subset of X and introduce the following definitions.

(1.1) $F(E) = F(E; f) = \int_E f(x)dx$.

(1.2) $B_*(\alpha) \equiv B_*(\alpha; f)$ *and* $B^*(\alpha) \equiv B^*(\alpha; f)$ *are, respectively, the greatest lower and least upper bounds of* $F(E)$ *taken over all sets* E *of measure* $|E| = \alpha$ $(0 \leqq \alpha \leqq 1)$.

Regarded as a function of E, $F(E)$ is called a generalized indefinite integral of $f(x)$ [**4**] or simply an indefinite integral of $f(x)$ [**1**]. In this section we develop the principal properties of the functions $B_*(\alpha)$ and $B^*(\alpha)$, and then obtain as a main result (see (1.9)) the fact that the values of $F(E)$ for $|E| = \alpha$ comprise the closed interval from $B_*(\alpha)$ to $B^*(\alpha)$. This is an extension of the known fact that $F(E)$ assumes all values between its optimum bounds, where no restrictions are placed on the measures of the sets involved [**4**]. In the second section the results of the first are applied to the problem of defining a mean value for $F(E)$ as E ranges over the measurable subsets of X.

Received by the editors April 26, 1951.