A REMARK ON CHARACTERISTICS

M. H. MARTIN

The algebraic function $\psi(\sigma, \theta)$ defined by $\psi^3 + 3\sigma\psi - 3\theta = 0$ is a solution¹ to Tricomi's equation

$$\psi_{\sigma\sigma} + \sigma\psi_{\theta\theta} = 0.$$

If ψ be taken as the stream function for the flow of the Tricomi gas, the stream lines $\psi = \text{const.}$ in the (θ, σ) -plane (hodograph plane) constitute a one-parameter family of straight lines enveloping the semi-cubic parabola $9\theta^2 = -4\sigma^3$. This semi-cubic parabola is one of the characteristics of Tricomi's equation. Within the region $9\theta^2 < -4\sigma^3$ the solution is triple-valued and the solution surface $\psi = \psi(\theta, \sigma)$ consists of three sheets, two of which unite along curves which project onto the characteristic $9\theta^2 = -4\sigma^3$ in the (θ, σ) -plane. Thus this characteristic serves as the locus of branch points for the algebraic function $\psi(\theta, \sigma)$.

If Tricomi's equation be replaced by a linear equation

$$L(u) = a_{ij}u_{x_i x_j} + b_i u_{x_i} = 0$$

(the repeated indices denoting summation from 1 to n) for an unknown function u of n independent variables $x_1, \dots, x_n$, it is natural to ask the following question.

If a solution to $L(u) = 0$ is defined implicitly by

$$F(x_1, \dots, x_n, u) = 0,$$

will the envelope of the one-parameter family of $(n-1)$ -dimensional hypersurfaces $u = \text{const.}$ (assuming that the envelope exists) be a characteristic hypersurface?

To begin with we have

$$F_{x_i} + F_u u_{x_i} = 0, \quad F_{x_i x_j} + F_{u x_i} u_{x_j} + F_{u x_j} u_{x_i} + F_{uu} u_{x_i} u_{x_j} + F_u u_{x_i x_j} = 0,$$

from which we obtain²

$$L(F) + 2\Delta_1(F_u, u) + F_{uu}\Delta_1(u, u) = 0.$$

When this equation is multiplied through by F_u^2 , we find

Received by the editors May 28, 1951.

¹ For details on the relation of this solution to gas dynamics, see M. H. Martin and W. Thickstun, *An example of transonic flow for the Tricomi gas*, Proceedings of the Fourth Symposium on Applied Mathematics.

² By $\Delta_1(u, v)$ we mean Beltrami's first differential operator, i.e., $\Delta_1(u, v) = a_{ij}u_{x_i}v_{x_j}$.

$$F_u^2 L(F) - 2F_u \Delta_1(F_u, F) + F_{uu} \Delta_1(F, F) = 0.$$

We assume that the one-parameter family of $(n-1)$ -dimensional hypersurfaces $F(x_1, \dots, x_n, u) = 0$ has an envelope

$$\Phi(x_1, \dots, x_n) = F(x_1, \dots, x_n, u(x_1, \dots, x_n)) = 0,$$

the function $u(x_1, \dots, x_n)$ being defined implicitly by

$$F_u(x_1, \dots, x_n, u) = 0.$$

At a point of contact of a member of the family with the envelope, not only does $\Phi = 0$, but also

$$F_{x_i} = \Phi_{x_i}, \quad F_u = 0,$$

to imply, provided $F_{uu} \neq 0$,

$$\Delta_1(\Phi, \Phi) = 0.$$

THEOREM. *If $F(x_1, \dots, x_n, u) = 0$ defines implicitly a solution to $L(u) = 0$, and if the one-parameter family of $(n-1)$ -dimensional hypersurfaces $u = \text{const.}$ in the space of the independent variables has an envelope upon which $F_{uu} \neq 0$, the envelope is a characteristic hypersurface.³*

For a multi-valued solution of $L(u) = 0$, the locus of points in the space of independent variables at which exactly two of the branches unite must be a characteristic hypersurface.

UNIVERSITY OF MARYLAND

³ This theorem is contained in a theorem of Le Roux and Delassus as quoted by J. Hadamard in his book *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Paris, 1932, p. 102. [Referee's note.]