ORTHOGONALITY RELATION FOR FROBENIUS- AND QUASI-FROBENIUS-ALGEBRAS

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The celebrated orthogonality relation for the coefficients of the regular representation of a group was extended first to the modular case by Nesbitt, and then to Frobenius-algebras by the writer; the proof was reproduced in [4][1] together with a second proof. Another proof of this orthogonality, for the coefficients of the regular representations of Frobenius-algebras, and its interesting application to faithful representations were given by Brauer [1]. In the present note we propose a still different proof, and generalize the orthogonality to quasi-Frobenius-algebras.

1. **A class of automorphisms in a Frobenius-algebra.** Let $\mathfrak{A}$ be a Frobenius-algebra [1; 2; 3; 4] over a field Ω, and let

$$(1) \qquad (a_1, a_2, \cdots, a_n) \qquad \text{with } a_\sigma a_\tau = \sum_\iota \alpha_{\sigma\tau\iota} a_\iota$$

be its basis. Let

$$(2) \qquad P = \left(\sum {}_\iota \alpha_{\sigma\tau\iota} \mu_\iota \right)$$

be a nonsingular parastrophic matrix. Let

$$(3) \qquad L(x) = (\lambda_{\sigma\tau}(x)), \qquad R(x) = (\rho_{\sigma\tau}(x))$$

be the left and right regular representations defined by the basis (1). We have

$$(4) \qquad R(x)P = PL(x).$$

With $x = \sum \xi_\iota a_\iota \in \mathfrak{A}$ we put [2, II, §1]

$$(5) \qquad \overset{*}{x} = \sum \overset{*}{\xi_\iota} a_\iota, \qquad (\overset{*}{\xi_\iota}) = (P')^{-1} P(\xi_\iota).$$

In particular

$$(6) \qquad (\overset{*}{a_1}, \overset{*}{a_2}, \cdots, \overset{*}{a_n}) P^{-1} = (a_1, a_2, \cdots, a_n)(P')^{-1}.$$

$*$ is an automorphism of $\mathfrak{A}$, and we have further, as a counterpart to (4),

$$(7) \qquad R(x)P' = P'L(x^*).$$

Received by the editors April 9, 1951.

[1] Numbers in brackets refer to the references at the end of the paper.

Also

$$(x^*(a_1^*, a_2^*, \cdots, a_n^*)P^{-1} =) x^*(a_1, a_2, \cdots, a_n)(P')^{-1}$$

$$(8) \quad \begin{aligned} &= (a_1, a_2, \cdots, a_n)L(x^*)(P')^{-1} \\ &= (a_1, a_2, \cdots, a_n)(P')^{-1}R(x) \end{aligned}$$

$$(= (a_1^*, a_2^*, \cdots, a_n^*)P^{-1}R(x)),$$

$$(9) \quad \begin{aligned} ((P')^{-1}(a_\iota^*)x &=) P^{-1}(a_\iota)x = P^{-1}R(x)(a_\iota) \\ &= L(x)P^{-1}(a_\iota)(= L(x)(P')^{-1}(a_\iota^*)). \end{aligned}$$

Thus, if we put

$$(10) \quad (b_1, b_2, \cdots, b_n) = (a_1, a_2, \cdots, a_n)(P')^{-1} = (a_1^*, a_2^*, \cdots, a_n^*)P^{-1},$$

then

$$(11) \quad x^*(b_1, b_2, \cdots, b_n) = (b_1, b_2, \cdots, b_n)R(x),$$

$$(12) \quad (b_\iota)x = L(x)(b_\iota).$$

We call $(b_1, b_2, \cdots, b_n)$ *conjugate* to $(a_1, a_2, \cdots, a_n)$.

With an arbitrary representation $Z(x)$ of $\mathfrak{A}$, consider the matrix

$$(13) \quad \mathfrak{Z} = \sum Z(b_\iota)a_\iota$$

in $\mathfrak{A}$. We have

$$(14) \quad \begin{aligned} \mathfrak{Z}x &= \sum Z(b_\iota)a_\iota x = \sum_\iota Z(b_\iota) \sum_\tau \rho_{\iota\tau}(x)a_\tau \\ &= \sum_\tau Z\left(\sum_\iota b_\iota\rho_{\iota\tau}(x)\right) a_\tau = \sum Z(x^*b_\tau)a_\tau \\ &= \sum Z(x^*)Z(b_\tau)a_\tau = Z(x^*)\mathfrak{Z}. \end{aligned}$$

Similarly we have, by virtue of (12) instead of (11),

$$(15) \quad x\mathfrak{Z} = \mathfrak{Z}Z(x).$$

A different choice of nonsingular parastrophic matrix gives rise to an automorphism congruent to * modulo inner automorphisms. On the other hand, if we start with another basis $(a_1, a_2, \cdots, a_n)Q$ of $\mathfrak{A}$, then the parastrophic matrix $Q'PQ$ belonging to it gives the same automorphism *. Take the second basis such that the left regular representation defined by it assumes a reduced form

$$(16)\qquad \begin{pmatrix} U^{(1)}(x) & & & & & & & & \\ & \ddots & & & & & & & \\ & & U^{(1)}(x) & & & & & & \\ & & & U^{(2)}(x) & & & & & \\ & & & & \ddots & & & & \\ & & & & & U^{(2)}(x) & & & \\ & & & & & & U^{(k)}(x) & & \\ & & & & & & & \ddots & \\ & & & & & & & & U^{(k)}(x) \end{pmatrix} \begin{matrix} \left.\right\} f(1) \text{ times} \\ \left.\right\} f(2) \text{ times} \\ \left.\right\} f(k) \text{ times} \end{matrix},$$

$$U^{(\kappa)}(x) = \begin{pmatrix} G^{(\kappa)}(x) & & M^{(\kappa)}(x) \\ & \ddots & \\ & & F^{(\kappa)}(x) \end{pmatrix}$$

with directly indecomposable $U^{(\kappa)}$ and irreducible $G^{(\kappa)}$, $F^{(\kappa)}$. Here $G^{(\kappa)}(x)$ is equivalent to $F^{(\kappa)}(x^{(*^{-1})})$; cf. [**2**, II, §1]. For those κ for which $U^{(\kappa)}$ are not irreducible, we can assume that our basis is chosen so that $G^{(\kappa)}(x) = F^{(\kappa)}(x^{(*^{-1})})$. As for those κ for which the $U^{(\kappa)}$ are irreducible, we see easily that we can choose μ_τ in (2) suitably so that $F^{(\kappa)}(x) = F^{(\kappa)}(x^{(*^{-1})})$; those κ correspond to simple subalgebras of $\mathfrak{A}$ which are direct components of $\mathfrak{A}$, and these components together form a semisimple (hence certainly symmetric) algebra, and we have merely to choose our * such that it induces the identity automorphism on this subalgebra. Under these adjustments we have

$$(17)\qquad U^{(\kappa)}(x) = \begin{pmatrix} F^{(\kappa)}(x^{(*^{-1})}) & & M^{(\kappa)}(x) \\ & \ddots & \\ & & F^{(\kappa)}(x) \end{pmatrix}.$$

2. Orthogonality relation for Frobenius-algebras.

THEOREM 1. *If a representation $Z(x)$ of $\mathfrak{A}$ does not contain $U^{(\kappa)}(x)$, then $F^{(\kappa)}(z) = 0$ for every element z of the matrix $\mathfrak{Z}$ in* (13). (If a representation contains $U^{(\kappa)}$, then it contains $U^{(\kappa)}$ as a direct component; cf. [**3**, §2, Remark 3].)

For, let $\mathfrak{l}$ be the linear space spanned, not necessarily linearly independently, by the elements of the row of $\mathfrak{Z}$ containing our z. It is a left-ideal of $\mathfrak{A}$, as (15) shows. Suppose, contrary to our assertion, $F^{(\kappa)}(z) \neq 0$. Then $\mathfrak{l}$ contains a certain primitive idempotent element e which generates a left-ideal $\mathfrak{A}e$ defining $U^{(\kappa)}$. But $\mathfrak{l}$ is a homomorphic image of the representation module $\mathfrak{z}$ of $Z(x)$, again by (15). Hence $\mathfrak{A}e$ is a submodule of a homomorphic image of $\mathfrak{z}$, which means that Z contains $U^{(\kappa)}$, contrary to our assumption.

In particular,

THEOREM 1′. *If $\zeta(x)$ is a coefficient in the representation $U^{(\lambda)}$ and $\lambda\neq\kappa$, then* $F^{(\kappa)}(\sum\zeta(b_\iota)a_\iota)=\sum\zeta(b_\iota)F^{(\kappa)}(a_\iota)=0$.

Further, the part of $U^{(\kappa)}$ complementary to $F^{(\kappa)}$ forms a representation which certainly does not contain $U^{(\kappa)}$. The same is the case with the part complementary to $G^{(\kappa)}$. Thus we have the following theorem.

THEOREM 1″. *If $\zeta(x)$ is a coefficient in the representation $U^{(\kappa)}$ which is not in $M^{(\kappa)}$, then* $F^{(\kappa)}(\sum\zeta(b_\iota)a_\iota)=\sum\zeta(b_\iota)F^{(\kappa)}(a_\iota)=0$.

REMARK. In Theorem 1″ only the reduced form (16) of $U^{(\kappa)}$ is used, and the more specified form (17) of $U^{(\kappa)}$ is unnecessary. In Theorem 1′, $U^{(\kappa)}$ can of course be in any form, not necessarily reduced.

We now assume that the $F^{(\kappa)}$ are absolutely irreducible, or at least that a certain $F^{(\kappa)}$, which we deal with, is so. Denote the degree of $U^{(\kappa)}$ by $u(\kappa)$. Because of our assumption the degree of $F^{(\kappa)}$ is equal to the multiplicity $f(\kappa)$ of $U^{(\kappa)}$ in the left regular representation; cf. (16). Construct

$$\mathfrak{U}^{(\kappa)}=\sum U^{(\kappa)}(b_\iota)a_\iota. \tag{18}$$

It satisfies

$$\mathfrak{U}^{(\kappa)}x=U^{(\kappa)}(x^*)\mathfrak{U}^{(\kappa)},\qquad x\mathfrak{U}^{(\kappa)}=\mathfrak{U}^{(\kappa)}U^{(\kappa)}(x). \tag{19}$$

It is well known, and easy to see, that $u(\kappa)$ coefficients of the first row (or any one of the first $f(\kappa)$ rows) of $U^{(\kappa)}$ are linearly independent. (As a matter of fact, it is also well known that the coefficients in the first $f(\kappa)$ rows of $U^{(\kappa)}$ altogether are linearly independent. But this may be seen from the sequel and need not be assumed as known.) Hence the $u(\kappa)$ elements in the first row of $\mathfrak{U}^{(\kappa)}$ are linearly independent. They form, according to (19), a left-ideal $\mathfrak{l}_1^{(\kappa)}$ which defines $U^{(\kappa)}$. $\mathfrak{l}_1^{(\kappa)}$ is a direct component of $\mathfrak{A}$. Moreover, the first $u(\kappa)-f(\kappa)$ of our elements of the first row of $\mathfrak{U}^{(\kappa)}$ form a (unique) maximal left-subideal of $\mathfrak{l}_1^{(\kappa)}$, as the reduced form of $U^{(\kappa)}$ shows; this follows also readily from Theorems 1′, 1″. Here the maximal left-subideal is precisely the intersection of $\mathfrak{l}_1^{(\kappa)}$ with the radical $\mathfrak{N}$ of $\mathfrak{A}$. Similarly the second, the third, $\cdots$, the $f(\kappa)$th rows of $\mathfrak{U}^{(\kappa)}$ span left-ideals $\mathfrak{l}_2^{(\kappa)}, \mathfrak{l}_3^{(\kappa)}, \cdots, \mathfrak{l}_{f(\kappa)}^{(\kappa)}$ belonging to $U^{(\kappa)}$.

Analogously, the last $f(\kappa)$ columns (i.e., the $u(\kappa)-f(\kappa)+1$st, $u(\kappa)-f(\kappa)+2$nd, $\cdots$, $u(\kappa)$th columns) of $\mathfrak{U}^{(\kappa)}$ span right-ideals $\mathfrak{r}_1^{(\kappa)}, \mathfrak{r}_2^{(\kappa)}, \cdots, \mathfrak{r}_{f(\kappa)}^{(\kappa)}$; all belonging to the representation $U^{(\kappa)}(x^*)$.

Now, take a system $\{e_{ij}^{(\kappa)}\}$ of elements in $\mathfrak{A}$ such that

$$(20) \qquad F^{(\kappa)}(e_{ij}^{(\kappa)}) = \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} (1 \text{ at } (i, j)), \quad F^{(\lambda)}(e_{ij}^{(\kappa)}) = 0 \qquad (\lambda \neq \kappa).$$

The elements $e_{ij}^{(\kappa)}$ form a system of matric units modulo $\mathfrak{N}$. We have $\mathfrak{U}^{(\kappa)}e_{ij}^{(\kappa)} = U^{(\kappa)}(e_{ij}^{(\kappa)*})\mathfrak{U}^{(\kappa)}$. By virtue of (17), which we make use of for the first time, this gives

$$(21) \qquad \mathfrak{l}_1^{(\kappa)} e_{1i}^{(\kappa)} \equiv \mathfrak{l}_i^{(\kappa)} \bmod \mathfrak{N};$$

more precisely

$$(22) \qquad (\text{1st row of } \mathfrak{U}^{(\kappa)}) e_{1i}^{(\kappa)} \equiv (i\text{th row of } \mathfrak{U}^{(\kappa)}) \bmod \mathfrak{N}.$$

Similarly we have

$$(23) \qquad e_{i1}^{(\kappa)} \mathfrak{r}^{(\kappa)} \equiv \mathfrak{r}_1^{(\kappa)} \bmod \mathfrak{N}$$

and in fact

$$(24) \qquad \begin{aligned} e_{i1}^{(\kappa)}(u(\kappa) - f(\kappa) + \text{1st column of } \mathfrak{U}^{(\kappa)}) \\ \equiv (u(\kappa) - f(\kappa) + i\text{th column of } \mathfrak{U}^{(\kappa)}) \bmod \mathfrak{N}. \end{aligned}$$

Thus the $(1, u(\kappa)-f(\kappa)+1)$-element of $\mathfrak{U}^{(\kappa)}$ lies in $e_{11}^{(\kappa)}\mathfrak{A}e_{11}^{(\kappa)} \bmod \mathfrak{N}$, and is, therefore, congruent to $\xi^{(\kappa)}e_{11}^{(\kappa)} \bmod \mathfrak{N}$ with $\xi^{(\kappa)} \in \Omega$. The $(i, u(\kappa)-f(\kappa)+j)$-element of $\mathfrak{U}^{(\kappa)}$ is then congruent to $\xi^{(\kappa)}e_{ij}^{(\kappa)} \bmod \mathfrak{N}$, according to (22), (24). Its representation matrix in $F^{(\kappa)}$ is the matrix with $\xi^{(\kappa)}$ at (i, j) and 0's elsewhere. This means, if we put

$$(25) \qquad F^{(\kappa)}(x) = (\phi_{st}^{(\kappa)}(x))_{st}, \qquad M^{(\kappa)}(x) = (\mu_{ij}^{(\kappa)}(x))_{ij},$$

that

$$(26) \qquad \phi_{st}^{(\kappa)}\left(\sum_\iota \mu_{ij}^{(\kappa)}(b_\iota)a_\iota\right) = \sum_\iota \phi_{st}^{(\kappa)}(a_\iota)\mu_{ij}^{(\kappa)}(b_\iota) = \xi^{(\kappa)}\delta_{sj}\delta_{ti}$$

$$(\xi^{(\kappa)}(\in \Omega) \neq 0).$$

Of course the relations in Theorems 1, 1′, 1″ can be written as

$$(27) \qquad \sum_\iota \phi_{st}^{(\kappa)}(a_\iota)\zeta(b_\iota) = 0,$$

where $\zeta(x)$ is as in those theorems. Thus:

THEOREM 2. *Let $F^{(\kappa)}$ be absolutely irreducible. With* (25) *we have* *the orthogonality* (26). *If* $\zeta(x)$ *is as in Theorems* 1, 1′, 1″, *then we have* (27).

3. **Quasi-Frobenius-algebras.** Let now $\mathfrak{A}$ be a quasi-Frobenius-algebra [2; 4] with a basis

$$(28) \qquad (a_1, a_2, \cdots, a_n),$$

which defines the left and right regular representations $L(x)$, $R(x)$ of $\mathfrak{A}$. Let

$$(29) \qquad (u_1, u_2, \cdots, u_n) = (a_1, a_2, \cdots, a_n)Q$$

be a second, auxiliary basis of $\mathfrak{A}$ such that the left regular representation $L_1(x) = Q_1^{-1}L(x)Q_1$ defined by it has a decomposed form similar to (16). Let $U_1^{(\kappa)}$ be its indecomposable components; $U_1^{(\kappa)} \simeq U^{(\kappa)}$. The right regular representation contains each $U_1^{(\kappa)}$ with a certain multiplicity, say $g(\kappa)$, greater than 0 (and it consists of them only). Put

$$h(\kappa) = \min (f(\kappa), g(\kappa)).$$

There exists a right-ideal of $\mathfrak{A}$ which defines a representation containing (directly) each $U_1^{(\kappa)}$ exactly $h(\kappa)$ times. Let

$$(30) \qquad (w_1, w_2, \cdots, w_m) \qquad (m = \sum h(\kappa)u(\kappa))$$

be its basis such that the representation defined by it has $U_1^{(\kappa)}$ in their order of indices, that is, first $h(1)$ times $U_1^{(1)}$, then $h(2)$ times $U_1^{(2)}$, and so on. Augment it with $(f(\kappa)-h(\kappa))u(\kappa)$ 0's after the $h(\kappa)u(\kappa)$ w's belonging to $U_1^{(\kappa)}$ $(\kappa = 1, 2, \cdots, k)$. Thus we obtain a vector

$$(31) \qquad (v_1, v_2, \cdots, v_n),$$

with $\sum(f(\kappa)-h(\kappa))u(\kappa)$ dummies 0, satisfying

$$(32) \qquad (v_\iota)x = L_1(x)(v_\iota).$$

Put

$$(33) \qquad (b_1, b_2, \cdots, b_n) = (v_1, v_2, \cdots, v_n)Q_1';$$

(b_ι) does not, together with (v_ι), form a basis of $\mathfrak{A}$, in general. We have

$$(34) \qquad (b_\iota)x = Q_1L_1(x)Q_1^{-1}(b_\iota) = L(x)(b_\iota).$$

With a representation $Z(x)$ of $\mathfrak{A}$, define $\mathfrak{Z}$ as in (13) by means of this (b_ι) (and the basis (a_ι)). Then we obtain (15) in the same way as before.

Now, consider any left regular representation of $\mathfrak{A}$ decomposed and reduced in the form (16). We can prove quite similarly that:

THEOREM 3. *With the vector* (b_ι) *in* (33), *constructed as above, Theorems* 1, 1′, 1″ *are valid for the quasi-Frobenius-algebra* $\mathfrak{A}$.

Consider next the matrix

$$\mathfrak{Z}^* = \sum Z(a_\iota)b_\iota \tag{35}$$

in $\mathfrak{A}$. By means of (34) we see

$$\mathfrak{Z}^* x = Z(x)\mathfrak{Z}^*. \tag{36}$$

This leads us to the following theorem.

THEOREM 4. *Suppose that Z does not contain the component $V^{(\kappa)}$ of the right regular representation having $F^{(\kappa)}$ as its top constituent. Then $F^{(\kappa)}(\mathfrak{z}) = 0$ for any element $\mathfrak{z}$ of $\mathfrak{Z}^*$. Hence, also in our quasi-Frobenius-algebra generalization Theorem 3 of Theorems* (1 *and*)1′, 1″ *we may interchange* (a_ι) *and* (b_ι) *provided we consider* $V^{(\kappa)}$ *in place of* $U^{(\kappa)}$.

(For a Frobenius-algebra such a modification is rather meaningless.)

Now, in order to specify our basis (b_ι) further, we start with a (maximal) system $\{e_i^{(\kappa)}\}$ of primitive idempotent elements in $\mathfrak{A}$ such that $\mathfrak{A}e_1^{(\kappa)}$ defines $U_1^{(\kappa)}$ $(\simeq U^{(\kappa)})$. Let $\{c_{ij}^{(\kappa)}\}$ be, for each κ, a system of matric units with $c_{ii}^{(\kappa)} = e_i^{(\kappa)}$. We assume that our auxiliary basis (u_ι) ((29)) is taken in accord with the decomposition

$$\mathfrak{A} = \sum \mathfrak{A}e_i^{(\kappa)} \tag{37}$$

and is composed of the bases of $e_j^{(\lambda)}\mathfrak{A}e_i^{(\kappa)}$ such that each basis of $e_j^{(\lambda)}\mathfrak{A}e_i^{(\kappa)}$ is obtained from that of $e_1^{(\lambda)}\mathfrak{A}e_1^{(\kappa)}$ by the left- and right-multiplications of $c_{j1}^{(\lambda)}$, $c_{1i}^{(\kappa)}$. (If the $F^{(\kappa)}$ are absolutely irreducible, this means that (u_ι) is a so-called Cartan basis; we may perhaps call our basis a Cartan basis in the wider sense.) Put

$$E = \sum_{\kappa}\sum_{i=1}^{h(\kappa)} e_i^{(\kappa)}; \tag{38}$$

we can, and shall, take $E\mathfrak{A}$ as our right-ideal possessing the basis (w_ι) ((30)). Replace in (u_ι) those u's not belonging to $\mathfrak{A}_0 = E\mathfrak{A}E$ simply by 0's, to obtain a vector

$$(y_1, y_2, \cdots, y_n); \tag{39}$$

(y_ι) is essentially a basis of $\mathfrak{A}_0$, augmented by some 0's. For $x_0 \in \mathfrak{A}_0$ we have

$$x_0(y_1, y_2, \cdots, y_n) = (y_1, y_2, \cdots, y_n)L_1(x_0) \qquad (x_0 \in \mathfrak{A}_0) \tag{40}$$

with our left regular representation L_1 of $\mathfrak{A}$ defined by (u_ι). Also

$$(y_\iota)x_0 = R_1(x_0)(y_\iota) \qquad (x_0 \in \mathfrak{A}_0). \tag{41}$$

$\mathfrak{A}_0$ is a Frobenius-subalgebra of (the quasi-Frobenius-algebra) $\mathfrak{A}$. Let † be its automorphism defined by a nonsingular parastrophic matrix as in §1. Take a basis of $\mathfrak{A}_0$ conjugate (with respect to the same parastrophic matrix) to the basis consisting of the nonzero y_ι, and augment it with 0's at the same places where $(v_\iota)E$ has 0's, to obtain

$$(z_1, z_2, \cdots, z_n). \tag{42}$$

From the Cartan basis property of (u_ι), in the wider sense, and from the way we enlarged (w_ι) into (v_ι), we see easily (cf. also the argument below)

$$(z_\iota)x_0 = L_1(x_0)(z_\iota) \qquad (x_0 \in \mathfrak{A}_0). \tag{43}$$

It is also not difficult to see

$$\overset{\dagger}{x_0}(z_1, z_2, \cdots, z_n) = (z_1, z_2, \cdots, z_n)R_1(x_0) \qquad (x_0 \in \mathfrak{A}_0). \tag{44}$$

Each idempotent element $e_j^{(\lambda)}$ is represented in L_1 by a diagonal matrix having 1's and 0's on the diagonal, and different $e_j^{(\lambda)}$ have 1's on different places (on the diagonal). The representation matrix of $c_{ij}^{(\lambda)}$ has 1's at the intersections of those rows and columns where the matrices of $e_i^{(\lambda)}$ and $e_j^{(\lambda)}$ have 1's (and 0's elsewhere). From this form of L_1 and (43) it follows that (z_ι) consists of the basis of the modules $\mathfrak{A}_0 e_j^{(\lambda)}$ $(1 \leqq j \leqq h(\lambda))$ (and 0's) and the part corresponding to $\mathfrak{A}_0 e_j^{(\lambda)}$ is obtained from the part corresponding to $\mathfrak{A}_0 e_1^{(\lambda)}$ by the right-multiplication of $c_{1j}^{(\lambda)}$. Replace in (z_ι) those 0's, which are in the places we had basis elements of $E\mathfrak{A}(1-E)$ in (v_ι), by the basis elements of $E\mathfrak{A}c_{1i}^{(\lambda)} = \mathfrak{A}_0 e_1^{(\lambda)} c_{1i}^{(\lambda)}$, $i = h(\kappa)+1, \cdots, f(\lambda)$, obtained from those (in (z_ι)) of $\mathfrak{A}_0 e_1^{(\lambda)}$ by the (right-)multiplication of $c_{1i}^{(\lambda)}$. Then we obtain a vector which is essentially a basis of $E\mathfrak{A}$, augmented with some 0's. Denote it (v_ι). It satisfies (32) (which justifies our notation). To see that we have merely to verify the relation for the elements x of a form $x = c_{j1}^{(\lambda)} x_0 c_{1i}^{(\kappa)}$ $(x_0 \in e_1^{(\lambda)} \mathfrak{A} e_1^{(\kappa)})$. Thus it suffices to verify the relation for $x = c_{j1}^{(\lambda)}$, x_0, and $c_{1i}^{(\kappa)}$. But the relation is clear for these elements either because of the above construction of our (v_ι) and the structure of the representation matrices of $c_{ij}^{(\kappa)}$, mentioned above, or because of (43). So, we can employ this (v_ι) as our vector in (31). It satisfies, besides (32),

$$\overset{\dagger}{x_0}(v_1, v_2, \cdots, v_n) = (v_1, v_2, \cdots, v_n)R_1(x_0) \qquad (x_0 \in \mathfrak{A}_0), \tag{45}$$

as follows from (44) and our construction of (v_ι).

Define $(b_1, b_2, \cdots, b_n)$ from our $(v_1, v_2, \cdots, v_n)$ again by (33). Then we have, besides (34), (15), (36), which we have already observed, also

$$\overset{\dagger}{x_0}(b_1, b_2, \cdots, b_n) = (b_1, b_2, \cdots, b_n)R(x_0) \qquad (x_0 \in \mathfrak{A}_0), \tag{46}$$

since $R=(Q_1')^{-1}R_1Q_1'$. We obtain then

$$\mathfrak{Z}x_0 = Z(\overset{\dagger}{x_0})\mathfrak{Z} \qquad (x_0 \in \mathfrak{A}_0) \tag{47}$$

with $\mathfrak{Z}$ in (13), as our analogue to (14). With $\mathfrak{Z}^*$ in (35) we have, from (45), also

$$\overset{\dagger}{x_0}\mathfrak{Z}^* = \mathfrak{Z}^*Z(x_0) \qquad (x_0 \in \mathfrak{A}_0). \tag{48}$$

Consider, after all these preliminary constructions, again a right regular representation of $\mathfrak{A}$ in a decomposed form (16). It needs not, of course, be defined by our auxiliary basis (u_ι) ((29)), but we assume that it is taken such that there exists for each κ a system of elements $\{e_{ij}^{(\kappa)}\ (i, j=1, 2, \cdots, h(\kappa)\}$ in $\mathfrak{A}_0$ satisfying (20), where $F^{(\kappa)}$ denotes the last irreducible constituent of the directly indecomposable component $U^{(\kappa)}$ in our regular representation. We take it further such that the representation matrices $F^{(\kappa)}(x_0^{(\dagger^{-1})})$ and $G^{(\kappa)}(x_0)$ with $x_0 \in \mathfrak{A}_0$, of respective degrees $f(\kappa)$ and $g(\kappa)$, coincide with each other in their places (i, j) with $i, j \leqq h(\kappa)$. We also assume the absolute irreducibility of $F^{(\kappa)}$.

With such $U^{(\kappa)}$ and our above (b_ι), we again consider (18). We have

$$x\mathfrak{U}^{(\kappa)} = \mathfrak{U}^{(\kappa)}U^{(\kappa)}(x), \qquad \mathfrak{U}^{(\kappa)}x_0 = U^{(\kappa)}(\overset{\dagger}{x_0})\mathfrak{U}^{(\kappa)} \quad (x_0 \in \mathfrak{A}_0). \tag{49}$$

The first row of $\mathfrak{U}^{(\kappa)}$ spans a left-ideal defining $U^{(\kappa)}$, and its last $f(\kappa)$ elements do not belong to the radical $\mathfrak{N}$. The same is the case with each of the first $g(\kappa)$ rows of $\mathfrak{U}^{(\kappa)}$, and for $i \leqq h(\kappa)$ the ith row is congruent modulo $\mathfrak{N}$ to the first row multiplied on the right by $e_{1i}^{(\kappa)}$. Further, each of the last $f(\kappa)$ columns of $\mathfrak{U}^{(\kappa)}$ is congruent modulo $\mathfrak{N}$ to the first of them, multiplied on the left by $e_{i1}^{(\kappa)}$ with respective i; observe that x is arbitrary in the first half of (49). In particular, the $(1, u(\kappa)-f(\kappa)+1)$-element of $\mathfrak{U}^{(\kappa)}$ lies again in $e_{11}^{(\kappa)}\mathfrak{A}e_{11}^{(\kappa)}$ mod $\mathfrak{N}$, and is thus congruent to an element $\xi^{(\kappa)}e_{11}^{(\kappa)}$ $(\xi^{(\kappa)} \in \Omega)$ mod $\mathfrak{N}$. Here $\xi^{(\kappa)} \neq 0$ since the element is not contained in $\mathfrak{N}$. Further, for $i \leqq h(\kappa)$, $j \leqq f(\kappa)$ we see that the $(i, u(\kappa)-f(\kappa)+j)$-element is congruent to $\xi^{(\kappa)}e_{ij}^{(\kappa)}$ mod $\mathfrak{N}$. Hence

$$(50)\qquad \phi_{st}^{(\kappa)}\left(\sum_{\iota}\mu_{ij}^{(\kappa)}(b_\iota)a_\iota\right)=\sum_{\iota}\phi_{st}^{(\kappa)}(a_\iota)\mu_{ij}^{(\kappa)}(b_\iota)=\xi^{(\kappa)}\delta_{sj}\delta_{ti}\qquad (\xi^{(\kappa)}(\in\Omega)\neq 0)$$

for $i=1, 2, \cdots, h(\kappa)$ (and $s, t, j=1, 2, \cdots, f(\kappa)$). On the other hand, since $b_\iota\in E\mathfrak{A}$, for each ι, $G^{(\kappa)}(b_\iota)$ has 0's in its last $g(\kappa)-h(\kappa)$ rows. This shows that the left-ideals of $\mathfrak{A}$ formed by the $h(\kappa)+1$st, $\cdots$, $g(\kappa)$th rows of $\mathfrak{U}^{(\kappa)}$ are properly homomorphic to $\mathfrak{l}_1^{(\kappa)}$, the left-ideal formed by the first row. Thus $\sum_\iota \mu_{ij}^{(\kappa)}(b_\iota)a_\iota\in\mathfrak{N}$ for $i=h(\kappa)+1, \cdots, g(\kappa)$. Hence (50) holds for $i=1, 2, \cdots, g(\kappa)$ and $s, t, j=1, 2, \cdots, f(\kappa)$ without restriction.

THEOREM 5. *The orthogonality* (50) *holds for* $i=1, 2, \cdots, g(\kappa)$ *and* $s, t, j=1, 2, \cdots, f(\kappa)$, $F^{(\kappa)}$ *being absolutely irreducible.*

Again we may consider $\mathfrak{Z}^*$, $\mathfrak{U}^{*(\kappa)}$ instead of $\mathfrak{Z}$, $\mathfrak{U}^{(\kappa)}$. Observing (36), (48) we obtain the following theorem.

THEOREM 6. *We can interchange* (a_ι) *and* (b_ι) *in the orthogonality* (50) *of Theorem* 5 *provided we consider* (*the normized*) $V^{(\kappa)}$ *in place of* $U^{(\kappa)}$ (*cf. Theorem* 4); $\xi^{(\kappa)}$ *may differ from that of Theorem* 5.

In fact, the argument for the case $j=h(\kappa)+1, \cdots, f(\kappa)$ is simpler than the one we had above for $i=h(\kappa)+1, \cdots, g(\kappa)$. For, we have simply $\phi_{jt}^{(\kappa)}(b_\iota)=0$ for those j.

ILLUSTRATION. Since the above construction is somewhat complicated, it is perhaps useful to illustrate it by an example. Consider the (quasi-Frobenius-) algebra $\mathfrak{A}$ consisting of matrices

$$(51)\qquad \begin{pmatrix} U_1^{(1)} & & \\ & U_1^{(1)} & \\ & & U_1^{(2)}\end{pmatrix}\quad\left(U_1^{(1)}=\begin{pmatrix}\beta & \gamma_1 & \gamma_2\\ & \alpha_{11} & \alpha_{12}\\ & \alpha_{21} & \alpha_{22}\end{pmatrix},\quad U_1^{(2)}=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \delta_1\\ \alpha_{21} & \alpha_{22} & \delta_2\\ & & \beta\end{pmatrix}\right)$$

over a certain field Ω. Let $e_1^{(1)}$, $e_2^{(1)}$, c_{12}, c_{21}, $e^{(2)}$, d_1, d_2, d_1', d_2' be the elements of $\mathfrak{A}$ which have respectively α_{11}, α_{22}, α_{12}, α_{21}, β, γ_1, γ_2, δ_1, $\delta_2=1$ and other coefficients equal to 0.

$$(52)\qquad (u_\iota)=(d_1, e_1^{(1)}, c_{21}; d_2, c_{12}, e_2^{(1)}; d_1', d_2', e^{(2)})$$

forms a basis of $\mathfrak{A}$, and in fact (51) is the left regular representation

L_1 of $\mathfrak{A}$ defined by this basis (52). The right regular representation defined by the same basis is

$$(53)\qquad R_1 = \left(\begin{array}{ccc|ccc|ccc} \alpha_{11} & & & \alpha_{12} & & & & & \\ & \alpha_{11} & & & \alpha_{12} & & \delta_1 & & \\ & & \alpha_{11} & & & \alpha_{12} & & \delta_1 & \\ \hline \alpha_{21} & & & \alpha_{22} & & & & & \\ & \alpha_{21} & & & \alpha_{22} & & \delta_2 & & \\ & & \alpha_{21} & & & \alpha_{22} & & \delta_2 & \\ \hline & & & & & & \beta & & \\ & & & & & & & \beta & \\ \gamma_1 & & & \gamma_2 & & & & & \beta \end{array}\right).$$

We have $f(1)=2=g(2)$, $f(2)=1=g(1)$, and $h(1)=h(2)=1$, while $u(1)=u(2)=3$. The basis $(e^{(2)}, d_1, d_2)$ of the right-ideal $e^{(2)}\mathfrak{A}$ gives the representation $U_1^{(1)}$, and the bases $(e^{(1)}, c_{12}, d_1')$, $(c_{21}, e_2^{(1)}, d_2')$ of $e_1^{(1)}\mathfrak{A}$, $e_2^{(1)}\mathfrak{A}$ give the representation $U_1^{(2)}$. As (v_ι) we can take

$$(54)\qquad (v_\iota) = (e^{(2)}, d_1, d_2; 0, 0, 0; e_1^{(1)}, c_{12}, d_1').$$

Put $E=e_1^{(1)}+e^{(2)}$. Then the left and the right regular representations of $\mathfrak{A}_0=E\mathfrak{A}E$ defined by its basis $(d_1, e_1^{(1)}; d_1', e^{(2)})$ are

$$(55)\qquad \begin{pmatrix} \beta & \gamma & & \\ & \alpha & & \\ & & \alpha & \delta \\ & & & \beta \end{pmatrix}, \qquad \begin{pmatrix} \alpha & & & \\ & \alpha & \delta & \\ & & \beta & \\ \gamma & & & \beta \end{pmatrix}.$$

Its nonsingular parastrophic matrix P_0, corresponding to the same basis, is given by, for instance,

$$(56)\qquad P_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding automorphism $\dagger$ of $\mathfrak{A}_0$ is given by

$$(57)\quad (d_1, e_1^{(1)}, d_1', e^{(2)})^{\dagger} = (d_1, e_1^{(1)}, d_1', e^{(2)})(P_0')^{-1}P_0 = (d_1', e^{(2)}, d_1, e_1^{(1)}).$$

A conjugate basis is given by

$$(e^{(2)},\ d_1,\ e_1^{(1)},\ d_1'). \tag{58}$$

Now, $(v_\iota)E=(e^{(2)}, d_1, 0; 0, 0, 0; e_1^{(1)}, 0, d_1')$. Augmenting (58) with 0's as we have 0's in $(v_\iota)E$, we obtain

$$(z_\iota) = (e^{(2)},\ d_1,\ 0;\ 0,\ 0,\ 0;\ e_1^{(1)},\ 0,\ d_1'). \tag{59}$$

Since here $(z_\iota)=(v_\iota)E$, rather accidentally, it is clear that

$$(z_\iota)x_0(= (v_\iota)x_0E = L_1(x_0)(v_\iota)E) = L_1(x_0)(z_\iota). \tag{60}$$

Further

$$x_0(e^{(2)}, d_1, 0, 0, 0, 0, e_1^{(1)}, 0, d_1') \\ = (e^{(2)}, d_1, 0, 0, 0, 0, e_1^{(1)}, 0, d_1') \begin{pmatrix} \beta & & & & \\ & \beta & \gamma & & \\ & & \alpha & & \\ & & & 0 & \\ \delta & & & & \alpha \end{pmatrix}.$$

Hence

$$\overset{\dagger}{x}_0(e^{(2)}, d_1, 0, 0, 0, 0, e_1^{(1)}, 0, d_1') \\ = (e^{(2)}, d_1, 0, 0, 0, 0, e_1^{(1)}, 0, d_1') \begin{pmatrix} \alpha & & & & \\ & \alpha & \delta & & \\ & & \beta & & \\ & & & 0 & \\ \gamma & & & & \beta \end{pmatrix}.$$

Here the 3rd (4th, 5th, 6th) and 8th rows of the matrix in the right-hand side may be modified arbitrarily. Thus we have

$$\overset{\dagger}{x}_0(e^{(2)}, d_1, 0, 0, 0, 0, e_1^{(1)}, 0, d_1') \\ = (e^{(2)}, d_1, 0, 0, 0, 0, e_1^{(1)}, 0, d_1')R_1(x_0) \tag{61}$$

with R_1 as in (53). The construction described above gives from $(z_\iota) = (e^{(2)}, d_1, 0, 0, 0, 0, e_1^{(1)}, 0, d_1')$ the same old $(v_\iota)=(e^{(2)}, d_1, d_2, 0, 0, 0, e_1^{(1)}, c_{12}, d_1')$.

If we take our (u_ι) ((52)) also as (a_ι), then $(b_\iota)=(v_\iota)$ too. We may also take $U^{(\kappa)}=U_1^{(\kappa)}$, though such identifications decrease the value of orthogonality. The verification of the orthogonality relations is left to readers.

If we take, instead of the above P_0 (in (56)), the parastrophic matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{62}$$

then the corresponding automorphism $\dagger$ of $\mathfrak{A}_0$ is given by

$$(d_1, e_1^{(1)}, d_1', e^{(2)})^\dagger = (d_1', d_1 - d_1' + e^{(2)}, d_1, d_1' - d_1 + e_1^{(1)}). \tag{63}$$

The corresponding conjugate basis is

$$(e^{(2)} - d_1', d_1, e_1^{(1)} - d_1, d_1'). \tag{64}$$

Thus the new vector (z_ι) is

$$(z_\iota) = (e^{(2)} - d_1', d_1, 0, 0, 0, 0, e_1^{(1)} - d_1, 0, d_1'). \tag{65}$$

For this new (z_ι) we verify also

$$(z_\iota)x_0 = L_1(x_0)(z_\iota), \quad x_0(z_1, z_2, \cdots, z_9) = (z_1, z_2, \cdots, z_9)R_1(x_0) \quad (x_0 \in \mathfrak{A}_0). \tag{66}$$

REMARK. Our orthogonality relations, for the quasi-Frobenius-algebra $\mathfrak{A}$, are closely related to those of the Frobenius-subalgebra $\mathfrak{A}_0$. But they assert more than the latter do. Observe in particular that (b_ι) is essentially more than a basis of $\mathfrak{A}_0$.

REMARK. Some of our constructions and assertions can be generalized to the case where $\mathfrak{A}$ is not a quasi-Frobenius-algebra, but where the left and the right regular representations of $\mathfrak{A}$ contain certain common direct components.

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