AN EXTENSION OF MERCER'S THEOREM

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Mercer's summability theorem [1] has received the attention of many writers, who have extended and generalised it in various ways [2, 3, 4, 7, 8, 9 *inter alia*]. The original version is substantially as follows.

MERCER'S THEOREM. *If s_n is real, $\sigma_n = (s_1+s_2+\cdots+s_n)/n$, and*

$$s_n + q\sigma_n \to (1+q)s \qquad (n \to \infty)$$

where $q > -1$, then $s_n \to s$.

Some of the known generalisations [4, 7, 8, 9] replace σ_n by other kinds of summation means. We propose to show that σ_n may be replaced by the means of *any* regular Toeplitz method of summation; and even by those of any convergence-preserving method (if the limits $(1+q)s$ and s are left unspecified). Some reduction in the domain of values of q is not unexpected. No other restriction is required, however, than that s_n should be bounded; and this is, in the generality of Toeplitz methods, necessary for the existence of the means.

Mercer's theorem takes the form "summability implies convergence" (without Tauberian condition) when expressed in terms of the summation means $\tau_n = (s_n + q\sigma_n)/(1+q)$; this is the character of the very general "Mercerian" theorems of Pitt [8]. They refer to certain integral means, and to Hausdorff means not usually restricted to have the form of τ_n; so our theorem may not be closely related to them.

Agnew [6] gave three theorems of the same type, referring to Toeplitz means. We show that two of these together compose exactly a special case of our Corollary 1 on positive regular summation matrices, apart from our hypothesis of boundedness which Agnew shows to be unnecessary in his more special context. His other theorem of this kind is slightly more general, but only in that it permits the matrix to differ negligibly from one which is positive; it is included in our Corollary 2.

Our restrictions on q may perhaps be heavier than necessary, at any rate in the contexts of particular methods of summation. For instance, Hardy [2] showed that Mercer's original theorem with arithmetic means σ_n holds throughout the half-plane $\mathcal{R}q > -1$ (and nowhere

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else), whereas our Corollary 1 gives it only in the circle $|q|<1$. On the other hand, Rogosinski [9] illustrated Pitt's theorems [8] by deducing Mercer's theorem for Euler means σ_n in the same domain, $|q|<1$, as Corollary 1 requires.

Karamata [7] gave an elegant generalisation of Mercer's theorem which includes those of several other writers [1, 2, 3, 4]. It is not included in our theorem; for it gives Mercer's theorem throughout $\mathcal{R}q>-1$, which ours does not. Nor is our theorem included in Karamata's, which is of a much more special character; indeed, his summation means are simple enough to express s_n in terms of them by simple algebra. Partial generalisations of Karamata's theorem can be obtained from ours; but they are rather immediate applications of it, and less general than might have been hoped for, so we omit them.

Note added in proof (15 April, 1952). Further work [11] has led to a version of our theorem in which the restriction (6) on q is relaxed. The principle used has also permitted other developments, one a general converse of Mears's consistency theorem for absolute summability, another regarding omission of the hypothesis that (s_n) is bounded.

Statement of the theorem. We lead up to the theorem of this paper by way of two of its special cases, in order of increasing generality. These cases are of sufficient intrinsic interest to merit separate statements, and in them the shape of Mercer's original theorem is clearly seen.

All numbers which are not suffixes may be complex, unless otherwise specified.

COROLLARY 1. *If (c_{nk}) is a positive regular summation matrix, (s_n) a bounded sequence such that*

$$s_n + q\sum_{k=1}^{\infty} c_{nk}s_k \to (1+q)s$$

and $|q|<1$, then $s_n \to s$.

COROLLARY 2. *If (c_{nk}) is a regular summation matrix, (s_n) a bounded sequence such that*

$$s_n + q\sum_{k=1}^{\infty} c_{nk}s_k \to (1+q)s, \tag{1}$$

$$|q| < 1\Big/\left(\limsup_{n\to\infty}\sum_{k=1}^{\infty}|c_{nk}|\right)^2, \tag{2}$$

then $s_n \to s$.

It is clear that Corollary 1 is obtained from Corollary 2 by supposing that $c_{nk} \geqq 0$; and Corollary 2 from the theorem following by supposing that $c_k = 0$ and $c = 1$. In the latter deduction $\lim s_n$ is determined by equating $\lim \sum c_{nk} s_k$ to it in (1).

THEOREM. *If* (c_{nk}) *is a convergence-preserving matrix, so that the following limits exist and are finite:*

$$\lim_{n\to\infty} c_{nk} = c_k \qquad \textit{for each } k, \tag{3}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} c_{nk} = c, \qquad \limsup_{n\to\infty} \sum_{k=1}^{\infty} | c_{nk} | = C; \tag{4}$$

and if (s_n) *is a bounded sequence such that*

$$s_n + q \sum_{k=1}^{\infty} c_{nk} s_k \tag{5}$$

converges as $n \to \infty$, *and*

$$| q | < \left| c - \sum_{k=1}^{\infty} c_k \right| \Big/ \left(C - \sum_{k=1}^{\infty} | c_k | \right)^2, \tag{6}$$

then (s_n) *is convergent.* (*The right side of* (6) *is to be* ∞ *if its denominator is* 0, *although its numerator is then also* 0.)

It might be thought that this theorem is only superficially more general than Corollary 2, and deducible by applying the latter to the regular matrix (d_{nk}) defined in (12). We show later that such a proof needs a more restrictive condition than (6).

Proof of the theorem. Our chief instrument is the following lemma on the "oscillation" of a sequence. It is a generalisation of an inequality of W. A. Hurwitz [5].

LEMMA. *If* (c_{nk}) *satisfies* (3) *and* (4) *with*

$$c \neq \sum_{k=1}^{\infty} c_k, \tag{7}$$

(s_n) *is a bounded sequence and* (σ_n) *its transform:*

$$\sigma_n = \sum_{k=1}^{\infty} c_{nk} s_k, \tag{8}$$

then the "oscillation" of (σ_n) *is related to that of* (s_n) *by*

$$\limsup | \sigma_m - \sigma_n | \leqq K \limsup | s_m - s_n |, \tag{9}$$

where the upper limits are taken as min $(m, n)\to\infty$, *and K depends only on the matrix* (c_{nk}):

$$K = \left(C - \sum_{k=1}^{\infty} |c_k|\right)^2 \Big/ \left|c - \sum_{k=1}^{\infty} c_k\right|. \tag{10}$$

Hypothesis (4) ensures that $\sum c_{nk}$ is absolutely convergent, at least for each $n > n_0$. For such n the means (8) exist, since (s_n) is bounded. Thus, for any integers $p > 1$, $m > n_0$, and $n > n_0$,

$$\sigma_m - \sigma_n = \sum_{k=1}^{p-1} (c_{mk} - c_{nk})s_k + \sum_{\mu=p}^{\infty} c_{m\mu}s_\mu - \sum_{\nu=p}^{\infty} c_{n\nu}s_\nu,$$

$$\begin{aligned}\left(c - \sum_{h=1}^{p-1} c_h\right)(\sigma_m - \sigma_n) = {} & \left(c - \sum_{h=1}^{p-1} c_h\right)\sum_{k=1}^{p-1} (c_{mk} - c_{nk})s_k \\ & + \left(c - \sum_{\nu=1}^{p-1} c_\nu - \sum_{\nu=p}^{\infty} c_{n\nu}\right)\sum_{\mu=p}^{\infty} c_{m\mu}s_\mu \\ & - \left(c - \sum_{\mu=1}^{p-1} c_\mu - \sum_{\mu=p}^{\infty} c_{m\mu}\right)\sum_{\nu=p}^{\infty} c_{n\nu}s_\nu \\ & + \sum_{\mu=p}^{\infty}\sum_{\nu=p}^{\infty} c_{m\mu}c_{n\nu}(s_\mu - s_\nu),\end{aligned}$$

the double series being summed in either order since they are absolutely convergent.

For any $\epsilon > 0$ there is an integer p_ϵ such that $|s_\mu - s_\nu| < \omega + \epsilon$ if min $(\mu, \nu) > p_\epsilon$, where $\omega = \limsup |s_m - s_n|$, which is finite since (s_n) is bounded. Choose $p > p_\epsilon$; and let M be an upper bound of $|s_n|$. Then

$$\begin{aligned}\left|c - \sum_{h=1}^{p-1} c_h\right| \cdot |\sigma_m - \sigma_n| & \\ \leqq {} & \left|c - \sum_{h=1}^{p-1} c_h\right| M \sum_{k=1}^{p-1} |c_{mk} - c_{nk}| \\ & + \left(\left|c - \sum_{\nu=1}^{\infty} c_{n\nu}\right| + \left|\sum_{\nu=1}^{p-1} (c_{n\nu} - c_\nu)\right|\right) M \sum_{\mu=1}^{\infty} |c_{m\mu}| \\ & + \left(\left|c - \sum_{\mu=1}^{\infty} c_{m\mu}\right| + \left|\sum_{\mu=1}^{p-1} (c_{m\mu} - c_\mu)\right|\right) M \sum_{\nu=1}^{\infty} |c_{n\nu}| \\ & + \sum_{\mu=p}^{\infty}\sum_{\nu=p}^{\infty} |c_{m\mu}|\,|c_{n\nu}|\,(\omega + \epsilon).\end{aligned}$$

Keeping p fixed and letting min (m, n) tend to infinity, the first three lines on the right tend to zero, by (3) and (4). Thus

$$\begin{aligned}\left|c-\sum_{h=1}^{p-1}c_h\right|\cdot\limsup\left|\sigma_m-\sigma_n\right| &\leqq (\omega+\epsilon)\quad\limsup\left(\sum_{\mu=p}^{\infty}\left|c_{m\mu}\right|\cdot\sum_{\nu=p}^{\infty}\left|c_{n\nu}\right|\right)\\ &\leqq (\omega+\epsilon)\cdot\limsup_{m\to\infty}\sum_{\mu=p}^{\infty}\left|c_{m\mu}\right|\cdot\limsup_{n\to\infty}\sum_{\nu=p}^{\infty}\left|c_{n\nu}\right|\\ &=(\omega+\epsilon)\left(C-\sum_{\mu=1}^{p-1}\left|c_\mu\right|\right)\left(C-\sum_{\nu=1}^{p-1}\left|c_\nu\right|\right),\end{aligned}$$

again using (3) and (4). Since this inequality holds for each $p>p_\epsilon$, we have, letting $p\to\infty$,

$$\left|c-\sum_{h=1}^{\infty}c_h\right|\cdot\limsup\left|\sigma_m-\sigma_n\right|\leqq(\omega+\epsilon)\left(C-\sum_{k=1}^{\infty}\left|c_k\right|\right)^2,$$

the series $\sum c_h$ being absolutely convergent (Hardy [10, p. 43]). By (7) and (10) we have

$$\limsup\left|\sigma_m-\sigma_n\right|\leqq K(\omega+\epsilon),$$

which gives (9) since ϵ is arbitrary.

PROOF OF THEOREM. Let σ_n be defined as in (8), and

$$r_n=s_n+q\sigma_n.$$

Taking upper limits as min $(m, n)\to\infty$ in the inequalities

$$\left|s_m-s_n\right|-\left|r_m-r_n\right|\leqq\left|q(\sigma_m-\sigma_n)\right|\leqq\left|s_m-s_n\right|+\left|r_m-r_n\right|$$

and taking account of the assumed convergence of r_n by means of

$$\limsup\left|r_m-r_n\right|=0,$$

we obtain

$$\limsup\left|q(\sigma_m-\sigma_n)\right|=\limsup\left|s_m-s_n\right|.$$

Thus

$$\begin{aligned}\limsup\left|s_m-s_n\right|&=\left|q\right|\limsup\left|\sigma_m-\sigma_n\right|\\&\leqq\left|q\right|K\limsup\left|s_m-s_n\right|\end{aligned}$$

by the lemma, K being defined by (10); hence

$$(1-\left|q\right|K)\limsup\left|s_m-s_n\right|\leqq 0$$

and so, since $|q|K<1$ by (6), the desired conclusion follows:

$$\limsup |s_m - s_n| = 0.$$

In the exceptional case

$$C - \sum_{k=1}^{\infty} |c_k| = 0$$

the lemma is unnecessary. For, on account of the inequalities

$$\sum_{k=1}^{\infty} |c_k| \leqq \liminf_{n\to\infty} \sum_{k=1}^{\infty} |c_{nk}| \leqq \limsup_{n\to\infty} \sum_{k=1}^{\infty} |c_{nk}| = C \tag{11}$$

we must have, as $n\to\infty$,

$$\sum_{k=1}^{\infty} |c_{nk}| \to \sum_{k=1}^{\infty} |c_k|.$$

This, together with (3), ensures that the matrix (c_{nk}) converts all bounded sequences into convergent sequences (Hardy [10, p. 43–47]). Thus the second member of (5) converges as $n\to\infty$, and so s_n must also converge.

Proof of theorem via Corollary 2. As indicated after the statement of the theorem, we now show that some generality would have been lost if we had proved only Corollary 2 and then sought to deduce the full theorem from it.

Under the hypotheses of the theorem, excluding (6), $\sum c_k$ is absolutely convergent, by (11). Write

$$c^* = \sum_{k=1}^{\infty} c_k, \qquad d_{nk} = \frac{c_{nk} - c_k}{c - c^*}, \tag{12}$$

supposing further that $c^* \neq c$; then (d_{nk}) is regular. Also $\sum c_k s_k$ exists, since s_n is bounded, so that

$$s_n + q(c - c^*) \sum_{k=1}^{\infty} d_{nk}s_k = s_n + q\sum_{k=1}^{\infty} c_{nk}s_k - q\sum_{k=1}^{\infty} c_k s_k$$

converges as $n\to\infty$, by (5). We can then infer the convergence of s_n from Corollary 2 provided that

$$|q(c - c^*)| < 1\Big/\left(\limsup_{n\to\infty} \sum_{k=1}^{\infty} |d_{nk}|\right)^2,$$

that is,

$$(13) \qquad |q| < |c - c^*| \Big/ \left(\limsup_{n\to\infty} \sum_{k=1}^{\infty} |c_{nk} - c_k|\right)^2.$$

Now this condition is no less restrictive than (6), and may be more so. For

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} |c_{nk} - c_k| \geqq \limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} |c_{nk}| - \sum_{k=1}^{\infty} |c_k| \right) = C - \sum_{k=1}^{\infty} |c_k|,$$

and the last expression is not negative, by (11); so that the right side of (6) is not less than that of (13), and may exceed it.

Agnew's theorems. Agnew [6] gave three theorems of Mercerian type, two of which are substantially as follows:

THEOREM 3.2 (AGNEW). *If (a_{nk}) is a positive, regular, and triangular matrix, and $a_{nn} \geqq \theta > 1/2$ for all sufficiently large n, then*

$$(14) \qquad \sum_{k=1}^{n} a_{nk} s_k \to s \textit{ implies that } s_n \to s.$$

THEOREM 4.1 (AGNEW). *If (a_{nk}) is real, regular, and triangular, and $a_{nk} \leqq 0$ for all $k \neq n$, then* (14) *holds.*

We shall show that these are respectively equivalent to the cases $q > 0$ and $q < 0$ of:

COROLLARY 0. *If (c_{nk}) is positive, regular, and triangular,*

$$(15) \qquad s_n + q \sum_{k=1}^{n} c_{nk} s_k \to (1 + q)s,$$

and $-1 < q < 1$, then $s_n \to s$.

This is the special case of Corollary 1 in which q is real and (c_{nk}) is triangular, except that (s_n) is not assumed to be bounded. Agnew shows explicitly that this assumption is not needed in his theorems. It is, however, essential to Corollary 1, as the following example shows:

$$c_{n\,n+1} = 1, \qquad c_{nk} = 0 \quad (k \neq n + 1), \qquad s_n = (-1/q)^n.$$

Corollary 0 deduced from Agnew's theorems. Let $\delta_{nk} = 0$ except that $\delta_{nn} = 1$, and define a_{nk} from c_{nk} by

$$(16) \qquad a_{nk} = \frac{\delta_{nk} + q c_{nk}}{1 + q}.$$

Assuming that (c_{nk}) satisfies the hypotheses of Corollary 0, we infer that (a_{nk}) is real, regular, and triangular, and

$$\sum_{k=1}^{n} a_{nk}s_k \to s.$$

If $0<q<1$, we have $a_{nk}\geqq 0$ and $a_{nn}\geqq 1/(1+q)>1/2$, so that $s_n\to s$ by Theorem 3.2.

If $-1<q<0$, we have $a_{nk}\leqq 0$ for all $k\neq n$, and so $s_n\to s$ by Theorem 4.1.

Agnew's theorems deduced from Corollary 0. Assuming the hypotheses of Theorem 3.2 we define c_{nk} from a_{nk} by the transformation (16) inverted, with $q=(1/\theta)-1$. We may assume $\theta<1$, since the hypotheses are satisfied a fortiori if θ is reduced; and since also $\theta>1/2$, we have $0<q<1$. We find that all other hypotheses of Corollary 0 are satisfied, and deduce that $s_n\to s$ as required.

For Theorem 4.1 we proceed similarly except that the role of θ is played by μ, an upper bound of a_{nn} exceeding unity; this exists since

$$a_{nn} \leqq \sum_{k=1}^{n} | a_{nk} | = O(1).$$

Then Corollary 0 applies to (c_{nk}) with $q=(1/\mu)-1$, so that q lies between -1 and 0.

Agnew's Theorem 3.1 is only slightly more general than his Theorem 3.2. In fact his proof amounts to reducing the former to the latter and then proving the latter. The reduction consists in indicating, as is proved by Hurwitz [5], that, if both

$$\sum_{k=1}^{n} a_{nk} \to 1 \quad \text{and} \quad \sum_{k=1}^{n} | a_{nk} | \to 1,$$

then

$$a_{nk} = \alpha_{nk} + \beta_{nk} \tag{17}$$

where

$$\alpha_{nk} \geqq 0, \qquad \sum_{k=1}^{n} | \beta_{nk} | \to 0; \tag{18}$$

so that (a_{nk}) differs negligibly from a positive regular matrix (α_{nk}). Agnew also shows in a few lines that there is no need to assume that (s_n) is bounded. We now show that our Corollary 2 includes the rest of his Theorem 3.1, which we may state as follows:

If (a_{nk}) *is regular and triangular,* $|a_{nn}| \geqq \theta > 1/2$ *for all sufficiently large* n, *and*

$$\sum_{k=1}^{n} |a_{nk}| \to 1, \qquad \sum_{k=1}^{n} a_{nk}s_k \to s,$$

where (s_n) *is bounded, then* $s_n \to s$.

The hypotheses imply that $\theta \leqq 1$; we choose λ so that $1/2 < \lambda < \theta$, and let

$$c_{nk} = \frac{a_{nk} - \lambda\delta_{nk}}{1 - \lambda}.$$

Then (c_{nk}) is regular; and (1) holds with $q = (1/\lambda) - 1$, so that $0 < q < 1$. Also

$$\sum_{k=1}^{n} |c_{nk}| = \frac{1}{1-\lambda}\left(\sum_{k=1}^{n} |a_{nk}| - |a_{nn}| + |a_{nn} - \lambda|\right) \to 1, \tag{19}$$

as we shall prove in a moment; thus (2) is satisfied and the required conclusion follows from Corollary 2.

To prove (19) we have, by (17) and (18),

$$\alpha_{nn} = |\alpha_{nn}| \geqq |a_{nn}| - |\beta_{nn}| \geqq \theta - \sum_{k=1}^{n} |\beta_{nk}| > \lambda \tag{20}$$

for all n sufficiently large. Also $|a_{nn}|$ lies between $\alpha_{nn} \pm |\beta_{nn}|$, and similarly $|a_{nn} - \lambda|$ between $\alpha_{nn} - \lambda \pm |\beta_{nn}|$, using (20). Thus, for all n sufficiently large,

$$-\lambda - 2|\beta_{nn}| \leqq |a_{nn} - \lambda| - |a_{nn}| \leqq -\lambda + 2|\beta_{nn}|,$$

in which both extreme terms tend to $-\lambda$, by (18). From this (19) follows.

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A NOTE ON GENERALIZED TAUBERIAN THEOREMS. ADDENDUM

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Lemma 9 of my paper mentioned above, in Proceedings of the American Mathematical Society vol. 2 (1951) pp. 335–349, may be stated thus.

LEMMA. *If there exists a sequence* $\{\lambda_p\}$ *such that* $1<\lambda_p\to 1$ *as* $p\to\infty$ *and*

$$\liminf_{u\to\infty} \operatorname*{lower\ bound}_{u<u'<\lambda_p u} \{A(u')-A(u)\} = \begin{cases} \text{either} & o_L(\lambda_p-1), \\ \text{or} & o_L(\log\lambda_p), \end{cases} \qquad \text{as } p\to\infty, \tag{1}$$

then

$$\liminf_{u\to\infty} A(u) = \liminf_{u\to\infty} \frac{1}{u}\int_0^u A(x)dx, \tag{2}$$

$$\limsup_{u\to\infty} A(u) = \limsup_{u\to\infty} \frac{1}{u}\int_0^u A(x)dx. \tag{3}$$

Dr. T. Vijayaraghavan has kindly pointed out to me that the proof of (3) which I have merely indicated might require some clarification as follows.

Taking one of the alternatives of (1), say the first, we can show that it implies, for any $\lambda>1$,

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