A REMARK ON A MONOTONE SINGULAR FUNCTION

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1. In a former paper¹ I have dealt with a monotone singular function $\omega(t; \alpha, \beta)$, where $\alpha > \beta > 1$, β an integer, $0 \leq t < \infty$; and with a monotone jump-function $t = G(x; \alpha, \beta)$ ($0 \leq x < \infty$) such that $\omega(t) = x = G^{-1}$. These functions can be generalised. Let β be any real number greater than one, $0 \leq x < \infty$,

$$(1.1) \quad \begin{aligned} \gamma(x) = \gamma(x; \alpha, \beta) &= \sum_{m=-\infty}^{\infty} \alpha^{-m} [\beta^m x] = \sum_{m \geq -\log x / \log \beta} \alpha^{-m} [\beta^m x]; \\ c_{\alpha, \beta} &= \lim_{0 > \epsilon \rightarrow 0} 1/\gamma(1 - \epsilon), \end{aligned}$$

where $[u]$ denotes the greatest integer not exceeding u ($u \geq 0$), and set

$$(1.2) \quad \begin{aligned} (i) \quad t &= g(x) = c_{\alpha, \beta} \gamma(x), \\ (ii) \quad \Lambda(t) &= \Lambda(t; \alpha, \beta) = x = g^{-1}(t) \quad (0 \leq t < \infty). \end{aligned}$$

It is not difficult to show that, in the special case when β is an integer,

$$(1.3) \quad c_{\alpha, \beta} = \frac{(\alpha - \beta)(\alpha - 1)}{\alpha(\beta - 1)}; \quad g(x - 0) \equiv G(x), \quad \Lambda(t) = \omega(t) \quad (0 \leq x, t < \infty).$$

The function $\omega(t; \alpha, \beta)$ was a generalisation of the Cantor ternary set function ($\alpha = 3, \beta = 2$).² Thus the latter is representable in the simple form

$$(1.4) \quad \omega(t; 3, 2) = g^{-1}(t); \quad t = g(x) = 2 \sum_{m=-\infty}^{\infty} 3^{-m-1} [2^m x] \quad (0 \leq t, x < \infty).$$

Several of the results on $G(x)$ and $\omega(t)$ hold for the general case (compare (1.5), (2.1), (2.2) with previous results). Some of the proofs can be shortened owing to the concise form of $g(x)$: and the result (2.5) on which (2.2) is based is even sharper than the corresponding one obtained previously.

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¹ Trans. Amer. Math. Soc. vol. 67 (1949) pp. 433-450. The function $t = G(x)$ is $G(x) = (\alpha - \beta)(\beta - 1)^{-1} \sum_{n=0}^{\infty} a_{n, x} \alpha^{-n}$, where $a_{n, x}$ is the number of positive integers n which are not divisible by β and such that $n\beta^{-n} < x$; while $\omega(t) = x = G^{-1}(t)$.

² H. Lebesgue, *Leçons sur l'intégration*, 2d ed., Paris, 1928. E. Hille and J. D. Tarmarkin, Amer. Math. Monthly vol. 36 (1929) pp. 255-264.

Plainly $g(x)$ is discontinuous at the points $X_{\mu,\nu} = \mu\beta^{-\nu}$ ($\mu = 1, 2, \dots$; $\nu = 0, \pm 1, \pm 2, \dots$) which are dense everywhere in $(0, \infty)$. Hence³ $g(x)$ is a strictly increasing jump-function, $\Lambda(t)$ is monotone, continuous, constant on the intervals of an open set of measure a for $0 \leq t \leq a$; also

$$(1.5) \quad \begin{aligned} g(0) &= 0 = \Lambda(0), & g(1-) &= 1 = \Lambda(1); \\ g(x) &= \alpha g(x/\beta), & \Lambda(t) &= \beta \Lambda(t/\alpha). \end{aligned}$$

2. We shall show that $\Lambda(t)$ is subadditive:⁴

$$(2.1) \quad \Lambda(t + \tau) \leq \Lambda(t) + \Lambda(\tau) \quad (t \geq 0, \tau \geq 0);$$

and that, given $\beta_0 > 1$ and $M > 0$,⁵

$$(2.2) \quad g(x; \alpha, \beta) \rightarrow x, \quad \Lambda(t; \alpha, \beta) \rightarrow t$$

as $\alpha - \beta \rightarrow 0$, uniformly for $\alpha > \beta \geq \beta_0$, $0 \leq x \leq M$, $0 \leq t \leq g(M)$. Since $[u+v] \geq [u] + [v]$ ($u \geq 0, v \geq 0$), $g(x+y) \geq g(x) + g(y)$ ($x \geq 0, y \geq 0$) by (1.1). Consequently (2.1) is true. To prove (2.2) we take first $M < 1$. For $0 \leq x < 1$ we have $\gamma(x) = \sum_{i=1}^{\infty} \alpha^{-i} [\beta^i x]$; hence

$$(2.3) \quad \begin{aligned} \frac{x\beta}{\alpha - \beta} &= \sum_{i=1}^{\infty} \alpha^{-i} \beta^i x \geq \gamma(x) \geq \sum_{i=1}^{\infty} \alpha^{-i} (\beta^i x - 1) \\ &= \frac{x\beta}{\alpha - \beta} - \frac{1}{\alpha - 1}. \end{aligned}$$

Multiplying by $c_{\alpha,\beta}$, taking $x \rightarrow 1$, and observing that $g(1-) = 1$, we deduce that

$$(2.4) \quad 0 \leq \frac{\beta c_{\alpha,\beta}}{\alpha - \beta} - 1 \leq \frac{c_{\alpha,\beta}}{\alpha - 1} \leq \frac{\alpha - \beta}{\alpha(\beta - 1)}.$$

From (2.3) we infer further that, for $0 \leq x < 1$, $0 \leq \beta c_{\alpha,\beta} x / (\alpha - \beta) - g(x) \leq c_{\alpha,\beta} / (\alpha - 1)$. Hence, taking $\beta c_{\alpha,\beta} / (\alpha - \beta) - 1 = u$, we have $-\{c_{\alpha,\beta} / (\alpha - 1) - ux\} \leq g(x) - x \leq ux$,

$$(2.5) \quad |g(x) - x| \leq \max \left(u, \frac{c_{\alpha,\beta}}{\alpha - 1} \right) \leq \frac{\alpha - \beta}{\alpha(\beta - 1)} \quad (0 \leq x < 1).$$

Consequently (2.2) is true for $M < 1$. Let now $M \geq 1$, $n = [\log M / \log \beta] + 1$, $x' = x\beta^{-n}$; then $M < \beta^n \leq M\beta$, $0 \leq x' < 1$, n

³ H. Kober, loc. cit., cf. 1.1.

⁴ Loc. cit., 1.4; here a different method of proof is needed.

⁵ Loc. cit., 2.1.

$\leq \log M / \log \beta_0 + 1$, $g(x') < 1$, and by (1.5),

$$\begin{aligned} |g(x) - x| &= \left| g\left(\frac{x}{\beta^n}\right) - \frac{x}{\alpha^n} \right| \alpha^n \\ &\leq | \{g(x') - x'\} \beta^n + g(x')(\alpha^n - \beta^n) | \\ &\leq M\beta |g(x') - x'| + Mn2^{n-1}(\alpha - \beta) \quad (\beta_0 \leq \beta < \alpha \leq 2\beta). \end{aligned}$$

Combining this with (2.5) we complete the proof.

3. By a very elementary theorem, any continuous function $f(x)$ ($0 \leq x \leq 1$) is uniformly approximated by step-functions. By means of (2.2), we can find *explicit* step-functions of the kind required.⁶ Take, for instance,

$$\begin{aligned} f_n(x) &= f \left\{ \frac{(\alpha - 2)(\alpha - 1)}{\alpha} \sum_{m=1}^{n^2} \alpha^{-m} [2^m x] \right\} \quad (0 \leq x < 1); \\ f_n(1) &= \lim_{x \rightarrow 1} f_n(x), \end{aligned}$$

where $\alpha = 2 + 1/n$. Then $f_n(x)$ is a step-function, and $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, uniformly in $(0, 1)$. We omit the proof.

Another application of (2.2) is the approximation to functions by nearly analytic functions, i.e. by functions which in a domain are continuous and, at almost all points, analytic. It can be shown that *any continuous function $f(x, y)$ ($x^2 + y^2 \leq 1$) of two real variables is uniformly approximated by nearly analytic functions $H_n(z)$ ($|z| = |x + iy| \leq 1$; $n = 1, 2, \dots$).* Given further $\phi(z)$, analytic and with bounded derivative for $|z| < 1$, approximating $H_n(z)$'s can be constructed such that $H'_n(z) = \phi'(z)$ p.p. and that, if $f(x, y)$ satisfies a Lipschitz condition, every $H_n(z)$ satisfies a Lipschitz condition also. Plainly these results are considerably more general than those on the approximation to *analytic* functions which we have proved previously.⁷

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⁶ Cf. the result on approximation by functions with an *infinity* of jumps in loc. cit., end of 2.8.

⁷ Loc. cit., 2.7.