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THE RECIPROCAL OF A CONTINUED FRACTION

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Stieltjes [3, Chapter X],[1] and later Rogers [2], gave formulas by means of which the reciprocal continued fractions for continued fractions of a certain class may be determined. We give below a theorem which extends the class of continued fractions to which this reciprocal transformation is applicable;[2] moreover, the theorem is stated in terms of certain parameters which facilitate the transformation.

From results of [1] it is known that there is a unique, one-to-one correspondence between formal power series, $F(z) = 1 + \sum_1^\infty c_p/z^p$, and C-fractions,

$$(1) \qquad F(z) \sim 1 + \frac{a_1}{z^{\beta_1}} + \frac{a_2}{z^{\beta_2}} + \cdots + \frac{a_n}{z^{\beta_n}} + \cdots,$$

where the β_p are integers, $\beta_0 = 0$, $\beta_{p-1} + \beta_p > 0$, and $a_p \neq 0$, $p = 1, 2, \cdots, n$, or $p = 1, 2, \cdots$, according as the C-fraction does or does not terminate. The power series for the pth approximant of the C-fraction agrees with $F(z)$ for the first $(\sigma_{p-1} + \sigma_p)$ terms, where $\sigma_p = \beta_0 + \beta_1 + \cdots + \beta_p$; this property characterizes the correspondence (1).

For simplicity we put $a_{2n} = 0$ in case a C-fraction terminates with the $(2n-1)$th partial quotient. Our result is stated below for a class of terminating C-fractions. In the nonterminating case the conditions of the theorem must hold for every index, n; consequently, a statement of our result in this case is obtained by replacing n by ∞.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Another theorem on reciprocals of C-fractions was given by Evelyn Frank, Amer. J. Math. vol. 68 (1946) pp. 89–108. The domain of application of the two theorems does not overlap.

THEOREM. *The correspondence*

$$(2) \qquad F(z) \sim 1 + \frac{a_1}{z^{\beta_1}} + \frac{a_2}{1} + \cdots + \frac{a_{2n-1}}{z^{\beta_{2n-1}}} + \frac{a_{2n}}{1}$$

implies the correspondence

$$(3) \qquad \frac{1}{F(z)} \sim 1 + \frac{a_1^*}{z^{\beta_1}} + \frac{a_2^*}{1} + \cdots + \frac{a_{2n-1}^*}{z^{\beta_{2n-1}}} + \frac{a_{2n}^*}{1}$$

if and only if the correspondence (2) has the form

$$(4) \qquad F(z) \sim 1 + \frac{2g_1}{z^{\beta_1}} + \frac{g_1(g_2 - 1)}{1} + \cdots + \frac{(g_{2n-2} + 1)g_{2n-1}}{z^{\beta_{2n-1}}}$$
$$+ \frac{g_{2n-1}(g_{2n} - 1)}{1},$$

where $g_{2p-1} \neq 0$, $p = 1, 2, \cdots, n$, *and* $g_{2p}^2 \neq 1$, $p = 1, 2, \cdots, n-1$. *In this case (3) has the form*

$$(5) \qquad \frac{1}{F(z)} \sim 1 - \frac{2g_1}{z^{\beta_1}} + \frac{g_1(g_2 + 1)}{1} + \cdots + \frac{(g_{2n-2} - 1)g_{2n-1}}{z^{\beta_{2n-1}}}$$
$$+ \frac{g_{2n-1}(g_{2n} + 1)}{1},$$

which may be obtained from (4) by replacing g_p *by* $-g_p$.

We begin the proof by noting that the pth numerator, $A_p(z)$, and denominator, $B_p(z)$, of (2), which are the solutions of

$$(6) \qquad y_p = z^{\beta_p} y_{p-1} + a_p y_{p-2}, \qquad p = 1, 2, \cdots, 2n, \ \beta_{2p} = 0,$$

for which $y_{-1} = 1$, $y_0 = 1$, and $y_{-1} = 0$, $y_0 = 1$, respectively, are polynomials in z of degree σ_p with leading coefficient unity. Since

$$A_p(z)B_{p-1}(z) - A_{p-1}(z)B_p(z) = (-1)a_1 a_2 \cdots a_p, \qquad p = 1, 2, \cdots, 2n,$$

it follows that the pth approximant of (2), $A_p(z)/B_p(z)$, is irreducible. Also $B_{2p-1}(0) = 0$, $A_{2p-1}(0) \neq 0$, $p = 1, 2, \cdots, n$, while $B_{2p}(0) \neq 0$, $p = 1, 2, \cdots, n-1$, and $B_{2n}(0) = 0$ if and only if $a_{2n} = 0$. Similar remarks apply for the numerators, $A_p^*(z)$, and denominators, $B_p^*(z)$, of (3).

If (2) implies (3), then

$$F(z) \sim \frac{1}{1} + \frac{a_1^*}{z^{\beta_1}} + \frac{a_2^*}{1} + \cdots + \frac{a_{2n-1}^*}{z^{\beta_{2n-1}}} + \frac{a_{2n}^*}{1},$$

and the odd contraction of the continued fraction on the right gives

$$(7) \quad 1 - \cfrac{a_1^*}{z^{\beta_1} + a_1^* + a_2^*} \; \cfrac{a_2^* a_3^*}{-\; z^{\beta_3} + a_3^* + a_4^*} \; - \; \cdots \; \cfrac{a_{2n-2}^* a_{2n-1}^*}{-\; z^{\beta_{2n-1}} + a_{2n-1} + a_{2n}},$$

whose pth approximant is $B_{2p}^*(z)/A_{2p}^*(z)$. The even contraction of the C-fraction of (2) is

$$(8) \quad 1 + \cfrac{a_1}{z^{\beta_1} + a_2} \; \cfrac{a_2 a_3}{-\; z^{\beta_3} + a_3 + a_4} \; - \; \cdots \; \cfrac{a_{2n-2} a_{2n-1}}{-\; z^{\beta_{2n-1}} + a_{2n-1}^* + a_{2n}^*}$$

whose pth approximant is $A_{2p}(z)/B_{2p}(z)$, and from earlier remarks it follows that

$$(9) \qquad A_{2p}^*(z) = B_{2p}(z), \qquad B_{2p}^*(z) = A_{2p}(z), \qquad p = 1, 2, \cdots, n.$$

Since the continued fractions (7) and (8) are identical, we deduce the necessity of the conditions

$$(10) \qquad \begin{aligned} &a_1^* = -a_1, \qquad\qquad a_1^* + a_2^* = a_2, \\ &a_{2p}^* a_{2p+1}^* = a_{2p} a_{2p+1}, \qquad a_{2p+1}^* + a_{2p+2}^* = a_{2p+1} + a_{2p+2}, \\ &\qquad\qquad\qquad\qquad\qquad p = 1, 2, \cdots, n-1, \end{aligned}$$

which are precisely the formulas given by Rogers for the case $\beta_{2p-1} = 1$, $p = 1, 2, \cdots, n$, and are equivalent to the formulas of Stieltjes for this case.

Continuing with the proof, we note that $A_{2p-1}(0) \neq 0$, $p = 1, 2, \cdots, n$, $B_{2p}^*(0) \neq 0$, $p = 0, 1, \cdots, n-1$; hence by (9), $A_p(0) \neq 0$, $p = 0, 1, \cdots, 2n-1$, when (2) implies (3).[3] Then by (6)

$$a_{2p-1} = A_{2p-1}(0)/A_{2p-3}(0) = (g_{2p-2} + 1)g_{2p-1},$$
$$p = 2, 3, \cdots, n, \; a_1 = 2g_1,$$
$$a_{2p} = [A_{2p}(0) - A_{2p-1}(0)]/A_{2p-2}(0) = g_{2p-1}(g_{2p} - 1),$$
$$p = 1, 2, \cdots, n,$$

where

$$g_{2p-1} = A_{2p-1}(0)/2A_{2p-2}(0), \qquad g_{2p} = -1 + 2A_{2p}(0)/A_{2p-1}(0),$$
$$p = 1, 2, \cdots, n;$$

thus the correspondence (2) may be written in the form (4) where

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[3] When $\beta_{2p-1} = 1$ and the continued fractions are nonterminating, a theorem of H. S. Wall, Trans. Amer. Math. Soc. vol. 48 (1940) p. 168, is equivalent to the statement that (2) implies (3) if and only if $A_p(0) \neq 0$, $p = 1, 2, \cdots$.

$g_{2p-1} \neq 0$, $p = 1, 2, \cdots, n$, and $g_{2n}^2 \neq 1$, $p = 1, 2, \cdots, n-1$. It is now easy to show by induction that for these a_p, $p = 1, 2, \cdots, 2n$, the numbers a_p^* obtained from them by replacing g_p by $-g_p$, $p = 1, 2, \cdots, 2n$, are solutions of (10) for which $a_p^* \neq 0$, $p = 1, 2, \cdots, 2n-1$; consequently the correspondence (3) may be written in the form (5).

Conversely, if (4) holds, the partial numerators a_p and a_p^* of the continued fractions in (4) and (5) satisfy (10). Then the continued fractions (7) and (8) are identical and (9) holds, from which it is readily seen that the left member of the correspondence (5) is $1/F(z)$. Thus a correspondence (2) which can be written in the form (4) implies a correspondence of the form (3), and the proof of the theorem is complete.

It is readily found that the present theorem is applicable to a large class of continued fractions obtained from the continued fractions of Gauss and of Heine. For example the expansion of $F(a, -b, a; -x)/F(a, 1-b, 1+a; -x)$ is (cf. [4, p. 337])

$$
1 + \cfrac{\dfrac{a(a+b)}{a(a+1)} x}{1} + \cfrac{\dfrac{1 \cdot (1-b)}{(a+1)(a+2)} x}{1} + \cfrac{\dfrac{(a+1)(a+b+1)}{(a+2)(a+3)} x}{1}
$$
$$
+ \cfrac{\dfrac{2(2-b)}{(a+3)(a+4)} x}{1} + \cdots,
$$

which converges in the x-plane cut from -1 to $-\infty$, except perhaps for isolated points (cf. [4, p. 339]). The parameters of the theorem are found to be $g_{2p+1} = (a+b)/2(a+2n-1)$, $g_{2p} = 1 + 2p(p-b)/(a+b) \cdot (a+2p)$, $p = 1, 2, \cdots$, and application of the theorem with z replaced by $1/x$ yields the new expansion,

$$
1 - \cfrac{\dfrac{a+b}{a+1} x}{1} + \cfrac{\dfrac{(a+1)(a+b+1)}{(a+1)(a+2)} x}{1} + \cfrac{\dfrac{1 \cdot (1-b)}{(a+2)(a+3)} x}{1}
$$
$$
+ \cfrac{\dfrac{(a+2)(a+b+2)}{(a+3)(a+4)} x}{1} + \cfrac{\dfrac{2(2-b)}{(a+4)(a+5)} x}{1} + \cdots,
$$

for the reciprocal of the original series. It is easily shown that the new expansion converges in the x-plane cut from -1 to $-\infty$, except perhaps at isolated points.

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