INACCESSIBLE BOUNDARY POINTS

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1. **Introduction.** The Riemann mapping theorem gives immediately the existence of a function $F(z)$, analytic in E, the circle $|z|<1$, and mapping that region onto a region having inaccessible boundary points [**3**, pp. 179–200].[1]

However, so far as the author is aware, there are no known formal expressions for an $F(z)$ of this type, and it is the purpose of the present work to remedy this defect by giving a class of examples of such functions. As a particular case we shall see that the function

$$F(z) = \frac{z}{\prod_{n=1}^{\infty} (1 - z2 \cos (\pi/2^{(n+1)/2}) + z^2)^{1/2^n}} \tag{1.1}$$

maps E onto a region B_s of the w-plane, for which all of the points $w>8$ are inaccessible boundary points.

2. **The slit regions B_s.** We shall consider regions formed by deleting from the entire complex plane an infinite number of semi-infinite radial slits, symmetrically placed with respect to the real axis. Because of this symmetry it is sufficient to consider only the slits in the closed upper half-plane. Let S_n denote the slit with end point at $w=\rho_n e^{i\phi_n}$ and suppose the subscripts so chosen that

$$\pi = \phi_0 > \phi_1 > \phi_2 > \cdots > \phi_n > \cdots > \phi_\infty = 0. \tag{2.1}$$

Definition. A region B_s is said to be of type S if it is formed as described above, and if in addition:

(a) There is a constant $m>0$ such that

$$m \leqq \rho_n < \infty, \qquad n = 0, 1, 2, \cdots,$$

(b) $$\lim_{n\to\infty} \phi_n = 0,$$

(c) $$\rho_\infty = \liminf_{n\to\infty} \rho_n.$$

It is obvious that if $\rho_\infty < \infty$, then all of the points $w>\rho_\infty$ are inaccessible boundary points of B_s. Condition (a) assures the existence of an $F(z)$, such that $F(0)=0$, $F'(0)>0$, and $F(z)$ maps E conformally onto B_s. Theorem 1 gives somewhat more information about $F(z)$.

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[1] Numbers in brackets refer to the references at the end of the paper.

THEOREM 1. *For each region B_s of type S there are constants $c>0, \theta_n$,*

$$\pi > \theta_1 > \theta_2 > \cdots > \theta_n > \cdots > 0, \qquad \theta_n \to \theta_\infty = 0, \tag{2.2}$$

such that if

$$F(z) = \frac{cz}{\prod_{n=1}^{\infty} (1 - z2\cos\theta_n + z^2)^{\gamma_n}}, \tag{2.3}$$

where

$$\gamma_n \pi = \phi_{n-1} - \phi_n, \qquad n = 1, 2, \cdots, \tag{2.4}$$

then $F(z)$ maps E conformally onto B_s, with $F(0)=0$ and $F'(0)>0$. Conversely each function defined by (2.3) *and* (2.2) *with*

$$\sum_{n=1}^{\infty} \gamma_n = 1, \qquad \gamma_n > 0;\ n = 1, 2, \cdots, \tag{2.5}$$

maps E conformally onto a region of type S where the directions of the slits are determined by

$$\phi_n = \pi \sum_{j=n+1}^{\infty} \gamma_j. \tag{2.6}$$

PROOF. First note that each term of the product

$$P_n(z) \equiv (1 - z2\cos\theta_n + z^2)^{\gamma_n} = (1 - ze^{i\theta_n})^{\gamma_n}(1 - ze^{-i\theta_n})^{\gamma_n} \tag{2.7}$$

is to be understood as that branch of the function for which $P_n(0)=1$. Let $E_n(\delta)$ denote the region obtained by deleting from E the portion common to the circle $|z-e^{i\theta_n}| \leqq \delta$, and the portion common to the circle $|z-e^{-i\theta_n}| \leqq \delta$, $\delta>0$; and let $E(\delta)$ denote the intersection of the sets $E_n(\delta)$, $n=1, 2, \cdots$. Then it is clear from (2.7) that in $E_n(\delta)$

$$\delta^{2\gamma_n} \leqq |P_n(z)| \leqq 4^{\gamma_n}, \tag{2.8}$$

$$-\gamma_n\pi < \arg P_n(z) < \gamma_n\pi, \tag{2.9}$$

and hence for $0<\delta<1$,

$$|\log P_n(z)| \leqq \gamma_n\{\log 4 - 2\log\delta + \pi\}. \tag{2.10}$$

From (2.5) it follows that in $E(\delta)$ the series

$$\sum_{n=1}^{\infty} \log P_n(z) \tag{2.11}$$

converges uniformly and hence by the Weierstrass Theorem represents an analytic function in $E(\delta)$. Since δ can be taken arbitrarily small, the series (2.11) converges in E and hence the product in (2.3) is also convergent in the same region. In fact, the product is uniformly convergent on any closed arc of $|z|=1$ which is free of the points $e^{\pm i\theta_n}$, $n=1, 2, \cdots$.

Suppose now that we have a given fixed region B_s of type S. By the Riemann mapping theorem, there is a unique function $F_s(z)$ mapping E conformally on B_s, with $F_s(0)=0$ and $F_s'(0)>0$. From the symmetry of B_s, $F_s(z)$ is real on the real axis and further $rF_s(r)>0$ for $0<r^2<1$. Denote by E_c the region obtained by cutting the open unit circle along the negative real axis, $-1<r\leqq 0$. Then $u_s=\arg F_s(z)$ is harmonic in E_c, and further, with a suitable determination, $-\pi \leqq u_s \leqq \pi$. A consideration of $1/F_s(z)$ shows that each of the points on the inverted slits S_n^{-1} $(w=\rho e^{-i\theta_n},\ 0\leqq\rho\leqq\rho_n^{-1},\ 0<\phi\leqq\pi)$ is an accessible boundary point and, therefore, the zero of $1/F_s(z)$ corresponding to the vertex of the sector defined by S_n^{-1} and S_{n-1}^{-1} is the image of a well-determined point $z=e^{i\theta_n}$ [3, pp. 189–192]. Even more, the Schwarz reflection principle shows that $z=e^{i\theta_n}$ is a simple zero of $(F_s(z))^{-1/\gamma_n}$. Thus the function $F_s(z)$ determines a set of arcs on the boundary of the upper half of the unit circle with end points $e^{i\theta_n}$ satisfying (2.2), such that for $\theta_{n+1}<\theta<\theta_n$

$$\arg F_s(e^{i\theta}) = \sum_{j=n+1}^{\infty} \gamma_j\pi = \phi_n, \qquad n = 1, 2, \cdots.$$

For $\theta_1<\theta<\pi$, $\arg F_s(z)=\pi$. Finally, if $\theta_\infty>0$, then $\arg F_s(z)$ can be extended by continuity so that $\arg F_s(e^{i\theta})=0$ for $-\theta_\infty<\theta<\theta_\infty$. To see this last assertion, observe that any simple curve Γ_w in B_s joining $\rho_n e^{i\phi_n}$ and $\rho_n e^{-i\phi_n}$ is the image of some simple curve Γ_z in E joining $e^{i\theta_n}$ and $e^{-i\theta_n}$. Given $\epsilon>0$, an appropriately chosen Γ_w with n sufficiently large determines a curve Γ_z, which, together with the arc $z=e^{i\theta}$, $-\theta_n\leqq\theta\leqq\theta_n$, determines a region in which $|\arg F_s(z)|\leqq\phi_n<\epsilon$.

Thus $\arg F_s(z)$ is a harmonic function in E_c, continuous and bounded on the interior, constant on the boundary except for finite jumps of $\pm\gamma_n\pi$ at $\pm\theta_n$, and a jump of 2π at $z=0$.

Next, with the values of θ_n just determined for $F_s(z)$, and with the associated values of γ_n, form the function $F(z)$ as in (2.3). If $u \equiv \arg F(z)$ for z in E_c, then u is also harmonic there and

$$\begin{aligned} u &= \arg z - \sum_{j=1}^{\infty} \gamma_j \arg (1 - z2\cos\theta_j + z^2) \\ &= \sum_{j=1}^{\infty} \gamma_j\{\theta - \arg(1 - z2\cos\theta_j + z^2)\}. \end{aligned} \tag{2.12}$$

For $z=e^{i\theta}$,

$$(2.13)\quad \arg(1 - z2\cos\theta_j + z^2) = \begin{cases} \theta, & \text{if } 0 \leqq \theta < \theta_j, \\ \theta - \pi, & \text{if } \theta_j < \theta \leqq \pi. \end{cases}$$

Thus for the upper half of the boundary of E_c, u is a step function, such that for $\theta_{n+1} < \theta < \theta_n$

$$(2.14)\qquad u = \arg F(e^{i\theta}) = \sum_{j=n+1}^{\infty} \gamma_j\{\theta - (\theta - \pi)\} = \phi_n.$$

Similar results hold for the lower half of the boundary of E_c. Finally when $z=r$, $-1<r<0$, $u=\pm\pi$ according as z approaches the boundary from above or below.

Therefore $U=u_s-u$ is a bounded harmonic function, continuous in E_c and zero on the boundary except for an infinite number of points which have at most two limit points. The conformality of $F(z)$ and $F_s(z)$ at the origin permit us to remove the slit, and make the same assertions about U in E. It is then easy to see from the Poisson integral formula that $U\equiv 0$ [4, p. 321]. Thus $F(z)$ and $F_s(z)$ differ by at most a multiplicative positive constant. Thus if c is chosen properly in (2.3), $F(z)$ maps E onto the region B_s. It is now easy to see that $\theta_\infty = 0$, for otherwise the arc

$$z = e^{i\theta}, \qquad -\theta_\infty < \theta < \theta_\infty,$$

would go into a doubly covered slit on the real axis consisting of accessible boundary points. This is indeed a possibility, but we have excluded this possibility by condition (c) of the definition of the region B_s. If, in (2.3), $\theta_\infty > 0$ then $F(z)$ will map E onto a region with accessible boundary points on the positive real axis whether or not it has inaccessible boundary points. We have defined B_s in such a way as to exclude this occurrence in order to simplify the presentation.

The second part of Theorem 1 will be a trivial consequence of the preceding material as soon as we show that the accessible boundary points of B_s form a one-to-one image of the arc $|z|=1$, $z\neq 1$, under $F(z)$, i.e. as soon as we show that each slit S_j is doubly covered. Recalling (2.5) it is easy to see that

$$(2.15)\qquad \frac{zF'(z)}{F(z)} = \sum_{j=1}^{\infty} \frac{\gamma_j(1 - z^2)}{1 - z2\cos\theta_j + z^2}.$$

Thus $F'(-1)=0$, and if $z=1$ is a point of regularity of $F(z)$, i.e. if $\theta_\infty \neq 0$, then $F'(1)=0$. For $z=e^{i\theta}$, $\theta_{n+1}<\theta<\theta_n$,

$$(2.16)\quad \begin{aligned} \frac{zF'(z)}{F(z)} &= -i\sin\theta\left\{\sum_{j=1}^{n}\frac{\gamma_j}{\cos\theta-\cos\theta_j}-\sum_{j=n+1}^{\infty}\frac{\gamma_j}{\cos\theta_j-\cos\theta}\right\} \\ &= -i\sin\theta\{I(\theta)-D(\theta)\} \end{aligned}$$

where $I(\theta)$ is an increasing function, tending to ∞ as $\theta \to \theta_n$, and $D(\theta)$ is a decreasing function, decreasing from ∞ at $\theta = \theta_{n+1}$. Thus in each arc $\theta_{n+1} < \theta < \theta_n$, $n = 1, 2, \cdots$, $F'(z)$ has a simple zero which we denote by $z = e^{i\alpha_n}$, and $F(e^{i\alpha_n}) = \rho_n e^{i\phi_n}$, a slit end point.

In case $\theta_\infty > 0$, $z = 1$ furnishes the simple zero of $F'(z)$ for the arc $-\theta_\infty < \theta < \theta_\infty$, while $z = -1$ is always the simple zero for $\theta_1 < \theta < 2\pi - \theta_1$. The slits S_j are doubly covered, and thus Theorem 1 is proved.

We observe that if the region B_s is not assumed to be symmetrical, then (2.3) is replaced by

$$(2.17)\qquad F(z) = \frac{cz}{\prod_{n=1}^{\infty}(1 - ze^{i\theta_n})^{\gamma_n}},$$

where now

$$(2.18)\qquad \sum_{n=1}^{\infty}\gamma_n = 2, \qquad \gamma_n > 0.$$

If B_s has only m slits, then in (2.17) and (2.18) the product and sum have exactly m terms.[2] In the still more special case that the $e^{i\theta_n}$ are taken as the mth roots of unity, and the γ_n all equal, we obtain the well known

$$f(z) = \frac{cz}{(1 - z^m)^{2/m}},$$

which maps the unit circle on the w-plane with radial slits whose end points are the vertices of a regular m-gon.

Perhaps a more intuitive approach to (2.17) and (2.3) would have been through Alexander's theorem [1; 5, pp. 11–15], which states that if

$$(2.19)\qquad F(z) = zf'(z),$$

then $F(z)$ starlike with respect to the origin implies $f(z)$ convex, and conversely. Then (2.17) and (2.3) would be a simple consequence of

[2] After this paper was completed, the author learned from Professor Z. Nehari that formulae (2.17), (2.18), with a finite number of terms, were given earlier by C. Darwin [2].

the Schwarz-Christoffel transformation

$$f(z) = c\int_0^z \frac{dt}{\prod_{n=1}^{\infty}(1 - te^{i\theta_n})^{\gamma_n}}. \tag{2.20}$$

It is worth noting that even though $F(z)$ given by (2.17) may map E onto a region with inaccessible boundary points, the associated $f(z)$ given by (2.19) and (2.20) maps E onto a convex region which consequently has no inaccessible boundary points. Thus integration may destroy inaccessible boundary points.

3. Some examples of functions for which B_s has inaccessible boundary points. In general it may be very difficult to find the zeros of (2.16), but for the problem at hand this is not necessary, since if $\theta_{n+1} < \eta < \theta_n$, and if α_n is the zero of (2.16) corresponding to this interval, then $|F(e^{i\alpha_n})| \leqq |F(e^{i\eta})|$. This together with Theorem 1 gives the following results.

THEOREM 2. *A necessary and sufficient condition that $F(z)$ given by* (2.3) *maps E onto a region with inaccessible boundary points is that there is a constant M and a decreasing sequence $\{\eta_n\}$ such that*

$$\lim_{n\to\infty} \eta_n = 0, \tag{3.1}$$

and

$$|F(e^{i\eta_n})| \leqq M, \qquad n = 1, 2, \cdots. \tag{3.2}$$

Under these conditions all the points $\omega > M$ will be inaccessible boundary points.

To determine such a sequence we proceed thus. If, in (2.3), $c = 1$, then

$$\log |F(e^{i\eta})| = \log\frac{1}{2} - \sum_{j=1}^{\infty}\gamma_j \log |\cos\eta - \cos\theta_j|. \tag{3.3}$$

Since $\cos\eta - \cos\theta = -2\sin(\eta+\theta)/2\sin(\eta-\theta)/2$ and $2|x|/\pi \leqq |\sin x| \leqq |x|$ for $0 \leqq |x| \leqq \pi/2$, it follows that if $0 < \eta, \theta_j \leqq \pi/2$, then

$$\begin{aligned} Q(\eta) &\equiv \sum_{j=1}^{\infty}\gamma_j \log |\cos\eta - \cos\theta_j| \\ &\geqq \log\frac{2}{\pi^2} + \sum_{j=1}^{\infty}\gamma_j \log |\eta^2 - \theta_j^2|. \end{aligned} \tag{3.4}$$

Let us suppose that, given $\{\gamma_n\}$ satisfying (2.5), it is possible to determine an increasing sequence of positive constants $\{c_n\}$ such that the series

$$\sum_{j=1}^{\infty} e^{-c_j} = \sigma, \tag{3.5}$$

$$\sum_{j=1}^{\infty} \gamma_j c_j = M_1 \tag{3.6}$$

both converge. In $F(z)$ take $\theta_n > 0$ such that

$$\theta_n^2 = \frac{\pi^2}{4\sigma} \sum_{j=n}^{\infty} e^{-c_j}, \qquad n = 1, 2, \cdots, \tag{3.7}$$

and note that $\pi/2 = \theta_1 > \theta_2 > \cdots > \theta_\infty = 0$. Next set

$$\eta_n^2 = \frac{\theta_n^2 + \theta_{n+1}^2}{2} = \frac{\pi^2}{8\sigma}\left(e^{-c_n} + 2\sum_{j=n+1}^{\infty} e^{-c_j}\right) \tag{3.8}$$

and observe that $\theta_1 > \eta_1 > \theta_2 > \cdots > \theta_n > \eta_n > \cdots > 0$. Then for all $j, n = 1, 2 \cdots$,

$$\left| \eta_n^2 - \theta_j^2 \right| \geqq \frac{\pi^2}{8\sigma} e^{-c_j}. \tag{3.9}$$

Using this in (3.4), and taking into account (3.6), gives

$$Q(\eta_n) > -M_1 - \log 4\sigma. \tag{3.10}$$

Consequently from (3.3) and (3.4) we have

$$\left| F(e^{i\eta_n}) \right| < 2\sigma e^{M_1} = M. \tag{3.11}$$

We summarize these results as follows:

LEMMA 1. *Given $\{\gamma_n\}$ satisfying (2.5), if it is possible to find an increasing sequence of positive constants $\{c_n\}$ such that (3.5) and (3.6) are convergent series, then $F(z)$ defined by (2.3) with θ_n defined by (3.7) maps the unit circle onto a region B_s for which all the points $w > 2\sigma e^{M_1}$ are inaccessible boundary points.*

At first the author conjectured that for every convergent series (2.5) there is a sequence of positive constants $\{c_n\}$ such that (3.5) and (3.6) both converge. Professor James A. Jenkins refuted this conjecture with the following counter example. Set $\gamma_n = 1/n(\log n)^2$, $n \geqq 2$. If we could find $c_n > 0$ such that (3.5) and (3.6) both converge, then the sum

$$\sum_{2}^{\infty} (e^{-c_n} + \gamma_n c_n)$$

would also converge. But this last is impossible, since if $c_n \geqq \log n$, then $\gamma_n c_n \geqq 1/n \log n$, and if $c_n \leqq \log n$, then $e^{-c_n} \geqq 1/n$, so that in any case $e^{-c_n} + \gamma_n c_n > 1/n \log n$.

A rather general case in which Lemma 1 can be applied occurs when $\{\gamma_n\}$ is decreasing and $\sum \gamma_n \log n$ converges, for then it suffices to take $c_n = -\log \gamma_n$ so that (3.5) becomes (2.5). This is possible, for example, when $\gamma_n = kn^{-p}$, $k > 0$, $p > 1$.

A still simpler case occurs when $\{\gamma_n\}$ is a geometric sequence, i.e. $\gamma_n = A^{n-1}(1-A)$, $0 < A < 1$. In this case, taking $c_n = -\log A^{n-1}(1-A)$, we have, from (3.5), $\sigma = 1$, and from (3.6)

$$M_1 = -\frac{A}{1-A} \log A - \log^{*}(1-A), \tag{3.12}$$

so that

$$M = \frac{2}{(1-A)A^{A/(1-A)}}. \tag{3.13}$$

Equation (3.7) gives $\theta_n = \pi A^{(n-1)/2}/2$. This proves the following theorem.

THEOREM 3. *If* $0 < A < 1$, *the function*

$$F(z) = \frac{z}{\prod_{n=0}^{\infty} (1 - z2 \cos(\pi A^{n/2}/2) + z^2)^{A^n(1-A)}} \tag{3.14}$$

maps E onto a region of type S for which all the points $w > M$ are inaccessible boundary points, where M is given by (3.13).

The case $A = 1/2$ gives the function (1.1) mentioned in the introduction.

4. **Further examples.** By suitable combinations of $F(z)$ with certain simple functions, other interesting examples can be obtained. Thus if $f_c(z)$ maps onto E onto E_c, then $\log F(f_c(z))$ maps E onto a strip $|\Im(w)| < \pi$ cut by lines parallel to the real axis in such a way that the points $w > \log M$ are inaccessible boundary points. The function $(F^{1/2}(f_c(z)) - 1)/(F^{1/2}(f_c(z)) + 1)$ maps E onto the unit circle with circular arc slits so disposed that the points $(M^{1/2} - 1)/(M^{1/2} + 1) < w \leqq 1$ are inaccessible boundary points. It seems difficult however to give an expression for the function which maps E onto a circle

with radial slits and having inaccessible boundary points.

The inaccessible boundary points can be rotated by taking $F_k(z) = e^{i\delta_k}F(ze^{-i\delta_k})$, and it is then clear that

$$G_m(z) = \left\{ \prod_{k=1}^{m} F_k(z) \right\}^{1/m} \tag{4.1}$$

maps E onto a region having m radial lines with inaccessible boundary points, providing only that the $\theta_n^{(k)}$ for $F_k(z)$ satisfy suitable conditions.

Finally if

$$\frac{\pi}{2} > \theta_1 > \theta_2 > \cdots > \theta_\infty > \theta^* > 0, \tag{4.2}$$

$$\gamma^* + \sum_{n=1}^{\infty} \gamma_n = 1, \qquad \gamma^*, \gamma_n > 0, \tag{4.3}$$

and if the θ_n are suitably chosen, then

$$F(z) = \frac{z}{(1 - z2\cos\theta^* + z^2)^{\gamma^*} \prod_{n=1}^{\infty} (1 - z2\cos\theta_n + z^2)^{\gamma_n}} \tag{4.4}$$

maps E onto a region for which points on the two radial lines $w = \rho e^{\pm i\gamma^*\pi}$, $\rho > M$, are accessible from one side but inaccessible from the other.

References

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