RIGHT H^*-ALGEBRAS

M. F. SMILEY

A Banach algebra A is a *right H^*-algebra* in case A is a Hilbert space and for each x in A there is an x^* in A such that $(xy, z) = (x, zy^*)$ for every y and z in A, i.e., $R_y^* = R_{y^*}$. Ambrose[1] [1] determined the structure of A under the additional assumption that $L_x^* = L_{x^*}$. The purpose of this note is to offer modifications of some of the arguments of Ambrose which will yield the structure of right H^*-algebras which are *proper* in the sense that $Ax = 0$ implies that $x = 0$. Briefly, our result is that a proper right H^*-algebra is merely a proper H^*-algebra in which the norm has been changed to a certain equivalent norm in each of the simple components. As a consequence we observe that a proper right H^*-algebra is a dual ring in the sense of Kaplansky [2].

We are indebted first to H. T. Muhly whose question on the independence of the assumptions of Ambrose was the starting point of this investigation, and second to Kaplansky who pointed out to us that the continuity of the mapping $x \to x^*$ in a proper right H^*-algebra (which we had originally assumed) is a consequence of a very interesting result of C. E. Rickart [3, Lemma 5.3].

As an example of a right H^*-algebra, consider a set J, a fixed complex-valued function $\alpha(i, j)$ on $J \times J$ which satisfies

$$\alpha(j, i) = \bar{\alpha}(i, j), \tag{1}$$

$$\sum_i |x(i)|^2 \leqq \sum_{i,j} x(i)\alpha(i, j)\bar{x}(j) \leqq M \sum_i |x(i)|^2, \tag{2}$$

where M is a real number greater than 1 and $x(i)$ is a complex-valued function on J. The set A of all complex-valued functions $a(i, j)$ on $J \times J$ for which

$$\sum_{i,j,l} a(j, i)\alpha(j, l)\bar{a}(l, i) < +\infty \tag{3}$$

becomes a Hilbert space if we set

$$(a, b) = \sum_{i,j,l} a(j, i)\alpha(j, l)\bar{b}(l, i). \tag{4}$$

We complete the definition of A as in Example 1 of [1]. The inequalities (2) insure that multiplication is continuous and that the mapping $x \to x^*$ is onto A. We call this example a *matric* right H^*-

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[1] The numbers in brackets denote the references at the end of the paper.

algebra, and note that A will be an H^*-algebra if and only if all $\alpha(i, j)$ are equal (and that then our example is Example 1 of [1]). *We shall prove that every proper right H^*-algebra is a direct sum (in the sense of [1]) of matric right H^*-algebras.*

From now on let A be a proper right H^*-algebra. Then $x \to x^*$ is an involutorial anti-automorphism of A and $xx^*=0$ (or $x^*x=0$) implies that $x=0$. If R is a right ideal of A, so is R^P. In order to establish the same property for two-sided ideals of A, we have the following lemmas.

LEMMA 1. *If L is a left ideal of A and $Ax \subseteq L$, then $x \in L$.*

PROOF. Write $x=x_1+x_2$ with $x_1 \in L$, $x_2 \in L^P$. Then $zx=zx_1+zx_2$ and $zx_2 \in L$ for every z in A. Then $(zx_2, x_2)=0=(z, x_2x_2^*)$ for every z in A, and $x_2=0$, $x=x_1 \in L$.

LEMMA 2. *If I is a two-sided ideal of A, then $I^*=I$.*

PROOF. We have $xy=0$ for every $x \in I^P$ and $y \in I$ because I^P is a right ideal. Hence $(xy, z)=(x, zy^*)=0$ for every z in A and $Ay^* \subseteq I$, $y^* \in I$ by Lemma 1. Thus $I^* \subseteq I$ and $I=I^{**} \subseteq I^*$, $I^*=I$.

LEMMA 3. *If I is a two-sided ideal of A, so is I^P.*

PROOF. We have $(AI^P, I)=(A, I(I^P)^*)=(A, (I^PI^*)^*)=(A, 0)=0$, $AI^P \subseteq I^P$.

The proof of Theorem 3.1 of [1] shows that if $x \neq 0$ in A, then the subalgebras generated by x^*x and by xx^* each contain a nonzero *sa* idempotent. We shall call an idempotent $e \in A$ *w-primitive* in case e is not the sum of two doubly orthogonal *sa* idempotents.

LEMMA 4. *The sa idempotent e of A is w-primitive iff eA is a minimal right ideal of A.*

PROOF. Let R be a right ideal of A such that $0 \subset R \subset eA$. Then R contains a *sa* idempotent f and $e=\lambda f+g$ with $(f, g)=0$ and $\lambda \in C$. Hence $ef=f=\lambda f+gf$ and $(1-\lambda)f=gf$. But $(gf, f)=(g, f)=0$ so that $\lambda=1$ and $gf=0$. Since g is *sa*, $fg=0$, and $eg=g=g^2$. If $g=0$, then $e=f$ and $eA=fA \subseteq R$, a contradiction. We have proved that e is not w-primitive if Ae is not minimal. The converse is clear.

LEMMA 5. *Every sa idempotent e in A is a sum of a finite number of doubly orthogonal w-primitive sa idempotents which are in $eA \cap Ae$.*

PROOF. Assume, inductively, that $e=\sum_{i=1}^{n} f_i$, where f_i $(i=1, \cdots, n)$ are doubly orthogonal idempotents, and that $f_1=g_1+h_1$, where g_1 and h_1 are doubly orthogonal *sa* idempotents. Then

$f_1g_1 = g_1f_1 = g_1$ and $f_1h_1 = h_1f_1 = h_1$ so that, for $k \neq 1$, $f_kg_1 = f_kf_1g_1 = 0$, and likewise $g_1f_k = h_1f_k = f_kh_1 = 0$. Then also $(g_1, f_k) = (g_1, f_kg_1) = (h_1, f_k) = 0$, and $g_1, h_1, f_2, \cdots, f_n$ are doubly orthogonal *sa* idempotents. This process cannot go on forever since the norm of an idempotent is at least one. Since $ef_i = f_ie = f_i$, we see that $f_i \in eA \cap Ae$.

Now choose (using Zorn's lemma) a maximal set $\{e_j;\ j \in J\}$ of doubly orthogonal *w*-primitive *sa* idempotents of A. Let R be the closure of the algebraic union of the right ideals $R_j = e_jA$ for $j \in J$. Suppose that the left annihilator L of R is nonzero. Then L has a *sa* idempotent e and hence also a *w*-primitive *sa* idempotent g, by Lemma 5. But then $ge_j = 0$, $0 = (ge_j)^* = e_jg$, and $(g, e_j) = (g, e_jg) = 0$ for every $j \in J$. This contradicts the maximality of $\{e_j; j \in J\}$. Hence $L = 0$ and this is equivalent to $(AR^*)^P = 0$, $AR^* = A$, $RA = A \subseteq R$, $A = R$. (Cf. [2].)

We can now insert the word "right" before each of the H^*'s in the statement of Theorem 4.2 of [1]. We delete the first and third paragraphs of the proof given by Ambrose and add the following remarks: If $I(R_j) \neq I(R_i)$, then $I(R_i)I(R_j) = 0$, $0 = (I(R_i)I(R_j), A) = (I(R_i), AI(R_j))$, by Lemma 2. Thus $AI(R_j) \subseteq (I(R_i))^P$, and Lemma 1 yields $I(R_j) \subseteq (I(R_i))^P$. It follows that A is the direct sum (in the sense of [1]) of the simple right H^*-algebras $I(R_j)$.

Now apply the Pierce decomposition in a simple right H^*-algebra A and define matric units e_{ij} as in [1]. However, we have only that $(e_{ij}, e_{kl}) = 0$ for $j \neq l$ and $(e_{ji}, e_{li}) = (e_{jk}, e_{lk}) = \alpha(j, l) = \bar{\alpha}(l, j)$ for all $i, j, k, l \in J$. To see that A_{ij} span A, we first write[2] $x = \sum_j e_jx$, then $x_j^* = \sum_i e_iu_{ij}$, $x_j = \sum_i u_{ij}^*e_j^*$, $x = \sum_{ij} e_ju_{ij}^*e_i$, where we have used the continuity of multiplication and of the mapping $x \to x^*$. Then, as in [1], $x = \sum_{ij} a_{ij}e_{ij}$, where the a_{ij} are complex numbers uniquely determined by x. We easily find that (1), (3), and (4) hold. We then obtain the inequalities (2) from $\|xy\| \leqq \|x\|\,\|y\|$ and the continuity of the mapping $x \to x^*$. As in [1], it is now clear that A is isomorphic and isometric with a matric right H^*-algebra.

(Added June 23, 1952.) At the suggestion of the referee, we shall give an indication of our proof of (2). We first show that every finite principal submatrix of α has characteristic values which are greater than or equal to 1. Let $\sigma = (j_p;\ p = 1, \cdots, m)$ be a finite subset of J

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[2] (June 23, 1952.) The following neat proof of this expression is due to the referee. Since the involution is continuous, it suffices to show that $x = \sum_j xe_j$. We set $x_\sigma = \sum(xe_j; j \in \sigma)$ for a finite subset σ of J and compute $\|x_\sigma\|^2 = \sum_j \|xe_j\|^2; j \in \sigma)$ and $\|x - x_\sigma\|^2 = \|x\|^2 - \|x_\sigma\|^2 \geqq 0$. It is an easy consequence that $\sum_j xe_j$ converges unconditionally to an element y of A for which $(x - y)e_j = 0$ for every $j \in J$. But then $x - y = 0$, as desired.

and let α_σ be the corresponding principal submatrix of α. Then $\beta = u^*\alpha_\sigma u = \text{diag}\ (\beta_1, \cdots, \beta_m)$ for a suitable unitary matrix u. Let X_p be the m-rowed square matrix with 1 in the (p, p) position and zeros elsewhere $(p=1, \cdots, m)$. Then $\bar{u}X_pu^T$ is idempotent and the corresponding element f_p of A is also idempotent. It is also easy to see that $\|f_p\|^2 = \text{trace}\ (uX_pu^*\alpha_\sigma uX_pu^*) = \beta_p \geqq 1$ $(p=1, \cdots, m)$. Now let π_n $(n=1, 2, \cdots)$ be a sequence of finite subsets of J and suppose that the corresponding principal submatrices of α have characteristic values which are not bounded. Then, if we set $\sigma_n = \bigcup(\pi_m; m=1, \cdots, n)$, the principal submatrices α_n of α corresponding to σ_n will also have characteristic values which are not bounded. We then have $\beta_n = u_n^*\alpha_n u_n = \text{diag}\ (\beta_{1n}, \cdots, \beta_{k_n n})$ for suitable unitary matrices u_n, and by deleting certain of the σ_n, we may assume that $\beta_{k_n n} \geqq 2^{2n}$, while $1 \leqq \beta_{1n} \leqq s$ for all n and some fixed s. Let X_n be the k_n-rowed square matrix with $1/2^n$ in the $(1, k_n)$ position and zeros elsewhere, and let x_n in A correspond to $\bar{u}_nX_nu_n^T$. Then $\|x_n\|^2 = \text{trace}\ (X_n^T\beta_nX_n) = \beta_{1n}/2^{2n}$, while $\|x_n^*\|^2 = \text{trace}\ (X_n\beta_nX_n^T) = \beta_{k_n n}/2^{2n}$. Thus x_n approaches 0 while x_n^* does not approach zero and this is contrary to the continuity of the involution. Armed with these facts, it is an elementary exercise to complete the proof of (2).

References

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The State University of Iowa