A NOTE ON PAIRS OF NORMAL MATRICES WITH PROPERTY L

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The following is a generalization of a theorem due to Motzkin and Taussky [1] concerning matrices with property L. By definition, two matrices A and B have property L if any linear combination $\alpha A+\beta B$ (where α and β are complex numbers) has as characteristic roots the numbers $\alpha\lambda_i+\beta\mu_i$ where the λ_i are the characteristic roots of A and the μ_i are the characteristic roots of B both taken in a special ordering. It is shown in the above paper that if two hermitian matrices have property L, they commute.

The following lemma may be noted:

LEMMA. *If a normal matrix has its characteristic roots in the main diagonal, then the matrix is diagonal.*

Let A be normal with characteristic roots $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then there exists a unitary matrix U such that $UAU^{CT}=D$ where D is diagonal with the characteristic roots of A appearing in the diagonal. Then $UA^{CT}U^{CT}=D^{CT}$ and $UAA^{CT}U^{CT}=DD^{CT}$ and the characteristic roots of AA^{CT} are $|\alpha_i|^2$. Since the sum of the diagonal elements of AA^{CT} is equal to the sum of its characteristic roots, all nondiagonal elements of A must be zero if the characteristic roots of A appear along the diagonal of A.

At this point the following theorem (Theorem 1) from [1] is recalled: Let the n-rowed square matrices A and B have property L. Let t be the number of different characteristic roots of A and assume that all the characteristic roots λ_i of A are arranged in sets of equal roots. Let m_i be the multiplicity of the characteristic root λ_i of A and assume there are m_i independent characteristic vectors corresponding to each λ_i. Let $B_1=P^{-1}BP$ where $A_1=P^{-1}AP$ is in Jordan normal form. Then $B_1=(B_{ij})$, $i, j=1, \cdots, t$, where B_{ii} is an m_i-rowed square matrix ($i=1, \cdots, t$) and $|\lambda I-B_1| = \prod_{i=1}^{t}|\lambda I-B_{ii}|$.

THEOREM. *If A and B are normal matrices with property* L, *they commute.*

Let the matrix A be brought into diagonal form by an appropriate unitary matrix U such that $UAU^{CT}=D$ is diagonal where like roots are grouped together. Let

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$$UBU^{CT} = B_1 = (B_{ij})$$

$(i, j=1, \cdots, k)$ where each B_{ii} has an order equal to that of the number of like roots in the corresponding diagonal position in D. By the above-mentioned theorem the roots of B_1 are, in totality, the roots of $B_{11}, B_{22}, \cdots, B_{kk}$ taken together. From Schur [2] it is known that any square matrix with complex elements can be brought into a triangular matrix under a unitary transformation. Let U_i be a unitary matrix such that $U_iB_{ii}U_i^{CT}=T_i$ where each T_i is a triangular matrix. Let

$$V = \begin{bmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & U_k \end{bmatrix}.$$

Then $VDV^{CT}=D$ and VB_1V^{CT} is a normal matrix with the triangular matrices T_i in the main diagonal and consequently its characteristic roots down the main diagonal. Hence VB_1V^{CT} is diagonal because of the lemma and it follows that A and B commute.

Bibliography

1. T. S. Motzkin and O. Taussky, *Matrices with Property* L, Trans. Amer. Math. Soc. vol. 73 (1952) pp. 108–114.

2. I. Schur, *Über die characteristischen Wurzeln einer linearen Substitutionen mit einer Anwendung auf die Theorie der Integralgleichungen*, Math. Ann. vol. 66 (1909) pp. 488–510.

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