

It follows that the left member of (3.3) is positive for $k \geq 1$.

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FACTORIZATION OF n -SOLUBLE AND n -NILPOTENT GROUPS

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If n is an integer [positive or negative or 0], and if the elements x and y in the group G meet the requirements

$$(xy)^n = x^n y^n \quad \text{and} \quad (yx)^n = y^n x^n,$$

then we term the elements x and y n -commutative. It is not difficult to verify that n -commutativity and $(1-n)$ -commutativity are equivalent properties of the elements x and y , that (-1) -commutativity implies ordinary commutativity, and that commuting elements are n -commutative.

From any concept and property involving the fact that certain elements [or functions of elements] commute, one may derive new concepts and properties by substituting everywhere n -commutativity for the requirement of plain commutativity. This general principle may be illustrated by the following examples.

n -abelian groups are groups G such that $(xy)^n = x^n y^n$ for every x and y in G . They have first been discussed by F. Levi [3]; and they will play an important rôle in our discussion. Grün [2] has introduced the n -commutator subgroup. It is the smallest normal subgroup J of G such that G/J is n -abelian; and J may be generated by the totality of elements of the form $(xy)^n (x^n y^n)^{-1}$ with x and y in G . Dual to the n -commutator subgroup is the n -center. It is the totality of elements z in G such that $(zx)^n = z^n x^n$ and $(xz)^n = x^n z^n$ for every x in G ; see Baer [1] for a discussion of this concept.

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The concept of n -abelian group leads naturally to the concept of an n -soluble group as a group possessing a composition chain with n -abelian factors. Likewise one may generalize the concept of n -center to the concept of the upper n -center chain and this leads to the concept of n -nilpotent groups.

It is well known that finite abelian or nilpotent groups are direct products of their primary components; and Ph. Hall has shown that finite soluble groups may be represented as products of subgroups of all possible relatively prime orders. It is our objective here to obtain similar factorization theorems for n -abelian groups [Theorem A], n -soluble groups [Theorem B], and n -nilpotent groups [Theorem C].

We restrict our attention throughout to finite groups, since there the essential points of our problem may be brought out without conflict with the rather sophisticated problems of a totally different kind which arise when attempting to remove this finiteness hypothesis. Some indications as to the possibilities are given in the appendix.

1. Notations. We collect here a few concepts in the form most convenient for our discussion. All groups under consideration will be *finite*; the *order* of the group G will be denoted by $o(G)$ and the order of the element g will be denoted by $o(g)$.

Products of groups. The group G is the product of its subgroups U and V if every element in G may be represented in one and only one way in the form uv with u in U and v in V ; and this is equivalent to requiring that every element in G may be written in one and only one way in the form $v'u'$ with u' in U and v' in V . Clearly products of normal subgroups are just the direct products.

If in particular U and V are subgroups of relatively prime order of the group G and if $o(G) = o(U)o(V)$, then G is easily seen to be the product of its subgroups U and V .

It is clear how to define products of more than two factors.

n -elements and n -groups. The element g of the group G is said to be an n -element if $g^{n^i} = 1$ for some suitable positive integer i . This is equivalent to requiring that every prime divisor of $o(g)$ be a factor of n . If $o(t)$ is prime to n , then the element t in G is termed a Pn -element.

A group is an n -group [Pn -group] if all its elements are n -elements [Pn -elements]. The totality of n -elements in the group G is the n -component G_n of G ; and the Pn -component G_{Pn} is the totality of Pn -elements in G . Note that components are, in general, not subgroups.

n -commutativity. The elements x and y are n -commutative if

$$(1.1) \quad (xy)^n = x^n y^n \quad \text{and} \quad (yx)^n = y^n x^n.$$

It is known and easily verified that

(1.2) n -commutative elements are $(1-n)$ -commutative;

(1.3) n -commutativity of the elements x and y implies $x^{n-1}y^n = y^n x^{n-1}$.

See Baer [1, p. 173, Lemma 5].

2. n -abelian groups. The group G is termed n -abelian whenever any two of its elements are n -abelian; and this is equivalent to requiring

$$(xy)^n = x^n y^n \quad \text{for every } x \text{ and } y \text{ in } G.$$

It is clear that abelian groups are n -abelian and that n -abelian groups are $(1-n)$ -abelian [by (1.2)].

THEOREM A. *Every n -abelian group is the direct product of an n -group, a $(1-n)$ -group, and an abelian $Pn(1-n)$ -group.*

PROOF. If G is an n -abelian group, then we show first that

(a) the n -component G_n of G is a subgroup of G .

To prove this we note first that mapping the element x in G upon its n th power x^n constitutes an endomorphism of the n -abelian group G . Hence mapping x onto x^{n^i} is likewise an endomorphism of G whose kernel $K(n^i)$ is the totality of elements t in G such that $t^{n^i} = 1$. These normal subgroups $K(n^i)$ of G form an ascending chain, since $K(n^i) \leq K(n^{i+1})$; and it is clear that G_n is the set-theoretical join of these normal subgroups $K(n^i)$. Hence G_n is a normal subgroup of G , as we wanted to show.

(b) The $(1-n)$ -component G_{1-n} of G is a subgroup of G .

This follows from (a) once we recall that n -abelian groups are $(1-n)$ -abelian [by (1.2)].

(c) $Pn(1-n)$ -elements commute.

To see this consider $Pn(1-n)$ -elements x and y in G . Since the order of x is prime to n , we have $x = x'^n$ for suitable x' in G ; and since the order of y is prime to $n-1$, we have $y = y'^{n-1}$ for y' in G . Since G is n -abelian, we deduce now from (1.3) that

$$xy = x'^n y'^{n-1} = y'^{n-1} x'^n = yx,$$

as we claimed.

(d) The $Pn(1-n)$ -component $G_{Pn(1-n)}$ of G is an abelian subgroup of G .

This is an almost obvious consequence of (c).

Our Theorem A is an immediate consequence of (a), (b), and (d);

since components are fully invariant subgroups whenever they are subgroups.

COROLLARY. *n -abelian $Pn(1-n)$ -groups are abelian.*

This is an obvious consequence of Theorem A.

The reader should note the impossibility of proving decomposability or commutativity of the n -component of an n -abelian group. For every group G which satisfies $G^n = 1$ is n -abelian and such groups may have a great variety of structures. That this rather trivial example does not exhaust the possibilities in the least may be seen from the following remark: If the index of the center $Z(G)$ in G is n , then the transfer σ of G into $Z(G)$ maps every element in G upon its n th power. But the transfer is a homomorphism; and thus we see that a group G is n -abelian whenever $[G:Z(G)] = n$. For the necessary properties of the transfer, see Zassenhaus [4, pp. 131-132, in particular formulas (19) and (20)].

3. n -soluble groups. Following Grün [2] we define as the n -commutator subgroup $[G, G; n]$ of G the intersection of all the normal subgroups N of G such that G/N is n -abelian. Then $G/[G, G; n]$ is likewise n -abelian; and $[G, G; n]$ may be generated by the elements $(xy)^n(x^n y^n)^{-1}$ for x, y in G . It follows from (1.2) and the definition of the n -commutator subgroup that $[G, G; n] = [G, G; 1-n]$.

The n -derived series is now defined inductively by the rules:

$$G = G^{(0;n)}, G^{(i+1;n)} = [G^{(i;n)}, G^{(i;n)}; n].$$

It is clear that the n -derived series is a descending chain of fully invariant subgroups of the groups G ; and that $G^{(i;n)} = G^{(i;1-n)}$.

PROPOSITION. *The following properties of the group G are equivalent.*

- (i) *If M is a normal subgroup of G and $M < G$, then there exists an n -abelian normal subgroup, not 1, of G/M .*
- (ii) *G possesses a composition chain with n -abelian factors.*
- (iii) *G possesses a normal chain with n -abelian factors.*
- (iv) *The n -derived series terminates with 1.*

We omit the simple proof of this proposition, as we are not going to make much use of it.

If the group G satisfies the four equivalent conditions (i) to (iv) of this proposition, then G is termed n -soluble. It is clear that n -soluble groups are likewise $(1-n)$ -soluble and that subgroups and quotient groups of n -soluble groups are also n -soluble.

THEOREM B. *If the order $o(G)$ of the n -soluble group G is the product*

of relatively prime numbers h and k such that n as well as $1-n$ is prime to at least one of the numbers h and k , then G is the product of subgroups of orders h and k respectively.

PROOF. Since h and k are relatively prime and $o(G) = hk$, it suffices to prove the existence of subgroups of orders h and k respectively; and for reasons of symmetry it suffices to establish the existence of a subgroup of order h .

We prove our theorem by complete induction with respect to the order of G . Since our theorem is certainly true for groups of order 1, and since quotient groups of n -soluble groups are n -soluble, we may assume the validity of our theorem for every proper quotient group of G .

Since G is n -soluble, there exists an n -abelian normal subgroup $N \neq 1$ of G . We denote by h' the g.c.d. of h and $o(G/N)$ and by k' the g.c.d. of k and $o(G/N)$. Then $o(G/N) = h'k'$, the numbers h' and k' are relatively prime, and n as well as $1-n$ is relatively prime to at least one of the numbers h' and k' . We apply the inductive hypothesis. Consequently there exists a subgroup H^* of order h' of G/N ; and there exists one and only one subgroup H' of G such that $N \leq H'$ and $H'/N = H^*$.

Let $h = h'h''$ and $k = k'k''$. Then h'' and k'' are well determined relatively prime integers such that $o(N) = h''k''$ and such that n as well as $1-n$ is relatively prime to at least one of the numbers h'' and k'' . Since N is n -abelian, it follows from Theorem A that N is the direct product of its n -component N_n , its $(1-n)$ -component N_{1-n} , and of an abelian $Pn(1-n)$ -group M . The latter group M is the direct product of its primary components; and now one sees without difficulty that N is the direct product of uniquely determined subgroups H'' and K'' of orders h'' and k'' respectively. Since H'' and K'' are fully invariant subgroups of N , as are all components, H'' and K'' are normal subgroups of G . This implies in particular that K'' is a normal subgroup of H' . The orders of H'/K'' and of K'' are the relatively prime integers h and k'' respectively; and now we deduce from Schur's Splitting Theorem the existence of a subgroup H of H' such that $H' = HK''$ and $1 = H \cap K''$; see, for instance, Zassenhaus [4, p. 125, Theorem 25]. Since H and H'/K'' are isomorphic, H has order h ; and this completes the proof.

COROLLARY 1. *Every n -soluble group is the product of an n -group, a $(1-n)$ -group, and a soluble $Pn(1-n)$ -group.*

PROOF. It is a consequence of Theorem B that the n -soluble group G is the product of an $n(1-n)$ -group T and a $Pn(1-n)$ -group S . The

$n(1-n)$ -group T is n -soluble as a subgroup of the n -soluble group G ; and it follows from Theorem B that T is the product of an n -group U and a $(1-n)$ -group V . It is clear that G is the product of U , V , S . Finally S is n -soluble as a subgroup of the n -soluble group G . The factors of a composition chain of S are n -abelian $Pn(1-n)$ -groups; and it follows from the Corollary to Theorem A that they are abelian. Consequently S is soluble, as we wanted to show.

COROLLARY 2. *If the order $o(G)$ of the n -soluble group G is the product of the relatively prime integers h and k , and if $n(1-n)$ is relatively prime to h or to k , then any two subgroups of order h of G are conjugate in G .*

PROOF. Since our proposition is certainly true for groups of order 1, we may make the inductive hypothesis that our proposition holds for every n -soluble group whose order is smaller than $o(G)$.

Suppose now that $o(G) = hk$, that h and k are relatively prime and that $n(1-n)$ is relatively prime to h or to k . Consider two subgroups H and H^* of order h of G . Since G is n -soluble, there exists an n -abelian normal subgroup $N \neq 1$ of G . Then G/N is an n -soluble group of order smaller than $o(G)$; and thus it follows from our inductive hypothesis that our proposition holds in G/N . We let $h = h'h''$ and $k = k'k''$ where the integers h' , h'' , k' , k'' are determined in such a way that $o(G/N) = h'k'$, $o(N) = h''k''$. It is clear that h' , k' are relatively prime, that $n(1-n)$ is prime to at least one of the numbers h' and k' , and that h'' and k'' are relatively prime, that $n(1-n)$ is prime to at least one of the numbers h'' and k'' . We apply Theorem A on the n -abelian group N and find that N is the direct product of groups H'' and K'' of orders h'' and k'' respectively. Next we note that the subgroups NH/N and NH^*/N of G/N have both order h' . We deduce from the inductive hypothesis the existence of an element g in G such that

$$NH = N(g^{-1}H^*g);$$

and we let $H^{**} = g^{-1}H^*g$.

Since NH has order hk'' , one verifies that

$$NH = K''H = K''H^{**} \quad \text{and} \quad 1 = K'' \cap H = K'' \cap H^{**}.$$

If $n(1-n)$ is prime to k , then K'' is abelian as an n -abelian $Pn(1-n)$ -group [Corollary to Theorem A]; and if $n(1-n)$ is not prime to k , then $n(1-n)$ is prime to h and the n -soluble groups H and H^* of order h are $Pn(1-n)$ -groups which implies their solubility [by Corollary 1]. Thus K'' is abelian or NH/K'' is soluble. Hence we may

apply a Theorem of Witt-Zassenhaus asserting that H and H^{**} are conjugate in NH ; see Zassenhaus [4, p. 126, Theorem 27]. Consequently there exists an element t in NH such that $H = t^{-1}H^{**}t = (gt)^{-1}H^*(gt)$. This completes the proof.

REMARK 1. Whether or not it would suffice to make in Corollary 2 the weaker hypothesis of Theorem B is an open question whose answer depends essentially on the solution of the corresponding problem whether the Theorem of Witt-Zassenhaus which we applied holds without any solubility hypothesis.

REMARK 2. The preceding results are obvious generalizations of Ph. Hall's theorems for ordinary soluble groups. Noting that n -groups need not be n -soluble, it does not seem possible to show that the property of Theorem A is characteristic for n -solubility.

4. n -nilpotent groups. The n -center $Z(G; n)$ of the group G is the totality of elements z in G with the property:

$$(4.1) \quad (zg)^n = z^n g^n \text{ and } (gz)^n = g^n z^n \text{ for every } g \text{ in } G.$$

It is easily seen that $Z(G; n)$ is a characteristic subgroup of G ; and it follows from (1.2) that

$$(4.2) \quad Z(G; n) = Z(G; 1 - n).$$

If we denote by U^i the subgroup of G which is generated by all the i th powers of elements in U and by $[U, G]$ the subgroup generated by all the commutators $[u, g] = u^{-1}g^{-1}ug$ with u in U and g in G , then we may restate the following well known and easily verified results.

(4.3) *If the normal subgroup N of G is part of the n -center of G , then*

(a) $N^n \cap N^{1-n} \leq Z(G)$;

(b) $[N, G]^{n(1-n)} = 1$;

(c) $G^n \cap G^{1-n}$ is part of the centralizer of N in G .

For proofs see Baer [1, pp. 173–174, Folgerung 2, 3, 4].

LEMMA 1. *If the normal subgroup N of G is part of the n -center of G , and if S is an n -abelian subgroup of G , then NS is an n -abelian subgroup of G .*

PROOF. Suppose that u, v are elements in N and that s, t are elements in S . Then svs^{-1} belongs to N ; and we deduce from (1.1), (4.1), and (1.3) that

$$\begin{aligned} [(us)(vt)]^n &= [u(sv)t]^n = u^n(sv)t^n = u^n[(svs^{-1})st]^n \\ &= u^n(sv s^{-1})^n(st)^n = u^n(sv^n s^{-1})s^n t^n \\ &= u^n s v^n s^{n-1} t^n = u^n s^n v^n t^n = (us)^n(vt)^n; \end{aligned}$$

and this shows that the elements in NS are n -commutative.

COROLLARY 1. $G = Z(G; n)$ whenever $G/Z(G; n)$ is cyclic.

PROOF. If $G/Z(G; n)$ is cyclic, then there exists a cyclic subgroup S of G such that $G = Z(G; n)S$. Cyclic groups are abelian and consequently n -abelian. Application of Lemma 1 shows that G is n -abelian or $G = Z(G; n)$.

The upper n -center chain $Z_i(G; n)$ of the group G is defined inductively by the following rules:

$$1 = Z_0(G; n), Z_{i+1}(G; n)/Z_i(G; n) = Z[G/Z_i(G; n); n].$$

Clearly these are characteristic subgroups of G ; and

$$Z_i(G; n) = Z_i(G; 1 - n).$$

LEMMA 2. If N is a normal subgroup of G such that $N \cap Z_i(G; n) \neq 1$, then $N \cap Z(G; n) \neq 1$.

PROOF. By hypothesis there exists a smallest integer k such that $N \cap Z_k(G; n) \neq 1$. It is clear that $0 < k$ and that $N \cap Z_{k-1}(G; n) = 1$. Consider now an element z in $N \cap Z_k(G; n)$. If g is any element in G , then it follows from the normality of N that

$$(zg)^n \equiv g^n \equiv z^n g^n \text{ modulo } N,$$

since $z \equiv 1$ modulo N ; and it follows from the definition of the upper n -center chain that

$$(zg)^n \equiv z^n g^n \text{ modulo } Z_{k-1}(G; n),$$

since z belongs to $Z_k(G; n)$. Thus we see that $(zg)^n(z^n g^n)^{-1}$ belongs to $N \cap Z_{k-1}(G; n) = 1$. Hence $(zg)^n(z^n g^n)^{-1} = 1$ or $(zg)^n = z^n g^n$ and $(gz)^n = g^n z^n$ is seen likewise. This proves that z belongs to $Z(G; n)$; and thus we have shown that $1 < N \cap Z_k(G; n) \leq N \cap Z(G; n)$, as we wanted to show.

The group G shall be termed n -nilpotent if its upper n -center chain terminates in G . Clearly n -nilpotent groups are also $(1 - n)$ -nilpotent. There exist many equivalent definitions of n -nilpotency, for instance in terms of the lower n -center chain. As we are not going to use them, we omit their discussion.

COROLLARY 2. Subgroups and quotient groups of n -nilpotent groups are n -nilpotent.

The fairly obvious argument may be omitted.

COROLLARY 3. *If $N \neq 1$ is a normal subgroup of the n -nilpotent group G , then $N \cap Z(G; n) \neq 1$.*

This is a fairly immediate consequence of Lemma 2.

LEMMA 3. *If G is n -nilpotent, if x is an n -element and y a Pn -element in G , then $xy = yx$.*

PROOF. Let $S = \{x, y\}$ and $C = [S, S]$. Then C is generated by $c = [x, y]$ and the elements conjugate to c in S . Thus $C = 1$ if, and only if, $c = 1$. Assume now by way of contradiction that $C \neq 1$. Then there exists a maximal normal subgroup M of S such that $M < C$. It is clear that C/M is a minimal normal subgroup of S/M .

We deduce from Corollary 2 that $S^* = S/M$ is an n -nilpotent group. It will be convenient to let $s^* = Ms$ for s in S so that $c^* = [x^*, y^*]$, $C^* = C/M$. From $C^* \neq 1$ and Corollary 3 we deduce now that $C^* \cap Z(G^*; n) \neq 1$; and this implies because of the minimality of the normal subgroup C^* of S^* that $C^* \cap Z(G^*; n) = C^*$ or $C^* \leq Z(G^*; n)$.

We interrupt our argument to verify the following simple fact which will be used several times.

(+) If $x^*c^* = c^*x^*$, then $o(c^*)$ is a factor of $o(x^*)$; and if $y^*c^* = c^*y^*$, then $o(c^*)$ is a factor of $o(y^*)$.

It is clear that it suffices to verify one of the facts. Assume therefore that $y^*c^* = c^*y^*$. Then we deduce from $y^{*-1}x^*y^* = x^*c^*$ that $y^{*-i}x^*y^{*i} = x^*c^{*i}$ holds for every i . Hence

$$x^* = y^{*-o(y^*)}x^*y^{*o(y^*)} = x^*c^{*o(y^*)} \quad \text{or} \quad c^{*o(y^*)} = 1$$

so that $o(c^*)$ divides $o(y^*)$.

We return to the main part of our argument. We form $[C^*, S^*]$. This is a normal subgroup of S^* which is part of C^* . Because of the minimality of C^* there arise only two possibilities: either $[C^*, S^*] = 1$ or $[C^*, S^*] = C^*$.

Assume first that $[C^*, S^*] = 1$. Then C^* is part of the center of S^* . Hence $x^*c^* = c^*x^*$ and $y^*c^* = c^*y^*$. It follows from (+) that $o(c^*)$ is a common factor of the orders $o(x^*)$ and $o(y^*)$. But $o(x^*)$ is a divisor of $o(x)$ and $o(y^*)$ is a factor of $o(y)$. Thus $o(c^*)$ is a common divisor of the relatively prime numbers $o(x)$ and $o(y)$. Hence $o(c^*) = 1$ so that $c^* = 1$ and this implies $C^* = 1$ which is impossible.

Next we consider the other possibility, namely $[C^*, S^*] = C^*$. Since the normal subgroup C^* of S^* is part of the n -center of S^* , it follows from (4.3, b) that

$$C^{*n(1-n)} = [C^*, S^*]^{n(1-n)} = 1.$$

Since C^* is part of the n -center of S^* , C^* is n -abelian. We have

shown just now that C^* is an $n(1-n)$ -group. It follows from Theorem A that C^* is the direct product of an n -group and a $(1-n)$ -group. These direct factors of C^* are the n -component and the $(1-n)$ -component of C^* and as such they are characteristic subgroups of C^* . But characteristic subgroups of a normal subgroup of S^* are normal subgroups of S^* . Now it follows from the minimality of C^* that one of these factors equals 1 and the other factor equals C^* . In other words: C^* is either an n -group or a $(1-n)$ -group.

Assume first that C^* is an n -group. Since C^* is part of the n -center, it follows from Lemma 1 that $\{C^*, y^*\}$ is n -abelian. Since $\{C^*, y^*\}/C^*$ is a cyclic Pn -group and C^* is an n -group, it follows from Theorem A that the n -abelian group $\{C^*, y^*\}$ is the direct product of C^* and $\{y^*\}$. This implies in particular that $c^*y^* = y^*c^*$; and it follows from (+) that the order of the n -element c^* is a divisor of the order of the Pn -element y^* . The order of c^* is therefore 1 so that $c^* = 1$ and $C^* = 1$. This is impossible.

We assume next that C^* is not an n -group. Then C^* is a $(1-n)$ -group. If we substitute now in the argument of the preceding paragraph of our proof everywhere x^* for y^* , then we find again that $c^* = 1$ and $C^* = 1$. Thus we have been led to a contradiction by assuming that $[S, S] \neq 1$. Hence $[S, S] = 1$ so that in particular $[x, y] = 1$ or $xy = yx$, as we wanted to show.

LEMMA 4. *If the $Pn(1-n)$ -group S is part of the n -center of G , then $S \leq Z(G)$.*

PROOF. Since the n -center is n -abelian, it follows from Theorem A that the n -center of G is the direct product of an $n(1-n)$ -group U and a $Pn(1-n)$ -group V . Since V is the $Pn(1-n)$ -component of $Z(G; n)$, it contains all the $Pn(1-n)$ -elements in $Z(G; n)$. Hence $S \leq V$. Since V is a characteristic subgroup of the characteristic subgroup $Z(G; n)$ of G , V is a characteristic subgroup of G . It follows from (4.3, b) that

$$[V, G]^{n(1-n)} = 1.$$

But V and its subgroup $[V, G]$ are $Pn(1-n)$ -groups; and thus it follows that $[V, G] = 1$. This is equivalent to saying that V is part of the center of G . The subgroup S of V is consequently part of the center of G .

COROLLARY 4: *An n -nilpotent $Pn(1-n)$ -group is nilpotent.*

This is an almost immediate consequence of Lemma 4 and the definition of n -nilpotency.

THEOREM C. *Every n -nilpotent group is the direct product of an n -group, a $(1-n)$ -group, and a nilpotent $Pn(1-n)$ -group.*

PROOF. It follows from (4.2) that the n -nilpotent group G is likewise $(1-n)$ -nilpotent. Since the n -center is always n -abelian, n -nilpotent groups are also n -soluble. Thus it follows from Corollary 1 to Theorem B that G is a product of an n -group A , a $(1-n)$ -group B , and a $Pn(1-n)$ -group C . It follows from Lemma 3 that every element in A commutes with every element in B or in C and that likewise every element in B commutes with every element in A or in C . Now it is clear that G is the direct product of A , B , and C and that therefore A is the n -component of G , B the $(1-n)$ -component, and C the $Pn(1-n)$ -component of G .

C is a subgroup of the n -nilpotent group G . The $Pn(1-n)$ -group C is therefore itself n -nilpotent. Now it follows from Corollary 4 that C is an ordinary nilpotent group.

COROLLARY 5. *If the group G is n -nilpotent, and if the greatest common divisor of n and $o(G)$ as well as the greatest common divisor of $1-n$ and $o(G)$ is a prime power, then G is nilpotent.*

PROOF. It follows from Theorem C that G is the direct product of an n -group A , a $(1-n)$ -group B , and a nilpotent $Pn(1-n)$ -group. It follows from our hypothesis that the orders of A and B are prime powers. Hence A and B are nilpotent groups; and G is nilpotent as a direct product of nilpotent groups.

Appendix: Extension of results to infinite groups. The careful reader will have observed that the assumption that all groups are finite has not been used in all places with equal force. As a matter of fact some theorems remain true if we omit this hypothesis whereas in other cases some slight changes in wording or argument are necessary. But in some cases the finiteness hypothesis is quite essential. We give a short survey of the situation.

One sees easily that the argument used in proving Theorem A may actually be used to prove the following result:

The elements of finite order in an n -abelian group form a subgroup which is the direct product of an n -group, a $(1-n)$ -group, and an abelian $Pn(1-n)$ -group.

All the results of §4 which precede Theorem C remain true without the finiteness hypothesis, provided we introduce the transfinite terms of the upper n -central chain [in the obvious manner]; but it should be observed that in the proof of Lemma 3 some more sophisticated arguments have to be employed to get around the missing finiteness.

But the proof of Theorem B cannot easily be extended to infinite groups, since Schur's Splitting Theorem which we employed rather forcefully does not possess a generalization to infinite groups that could be employed here. If finitely generated groups all of whose elements are of finite order were finite, then these results could be extended too—this applies also to Theorem C. But the celebrated Problem of Burnside is still unsolved.

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