

REFERENCES

1. Eckford Cohen, *Sums of an even number of squares in $GF[p^n, x]$* . II, Duke Math. J. vol. 14 (1947) pp. 543-557.
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CONGRUENCES CONNECTED WITH THREE-LINE LATIN RECTANGLES

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1. **Introduction.** In a recent paper [1], J. Riordan set up the recurrences

$$(1.1) \quad K_n = n^2 K_{n-1} + (n)_2 K_{n-2} + 2(n)_3 K_{n-3} + k_n,$$

where $(n)_r = n(n-1) \cdots (n-r+1)$, and

$$(1.2) \quad k_n + n k_{n-1} = -(n-1)2^n;$$

here $K_n = K(3, n)$, the number of reduced three-line latin rectangles. He also proved the congruences

$$(1.3) \quad k_{n+p} \equiv 2k_n, \quad K_{n+p} \equiv 2K_n \pmod{p},$$

where p is a prime > 2 .

In the present note we shall extend (1.3). We show first that for arbitrary m ,

$$(1.4) \quad k_{n+m} \equiv 2^m k_n, \quad K_{n+m} \equiv 2^m K_n \pmod{m}.$$

More generally if we define

$$(1.5) \quad \Delta f(n) = f(n+m) - 2^m f(n), \quad \Delta^r f(n) = \Delta \Delta^{r-1} f(n)$$

for fixed $m \geq 1$, then

$$(1.6) \quad \Delta^r k_n \equiv 0 \equiv \Delta^r K_n \pmod{m^r}$$

for all $r \geq 1$.

2. Proof of (1.4). In (1.2) replace n by $n+m$ so that

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$$k_{n+m} + (n+m)k_{n+m-1} = -(n+m-1)2^{n+m}.$$

Comparison with (1.2) yields

$$(k_{n+m} - 2^m k_n) + n(k_{n+m-1} - 2^m k_{n-1}) + m(k_{n+m-1} + 2^{n+m}) = 0$$

or more briefly, using (1.5),

$$(2.1) \quad \Delta k_n + n\Delta k_{n-1} + m(k_{n+m-1} + 2^{n+m}) = 0.$$

Clearly (2.1) implies

$$\Delta k_n \equiv (-1)^n n! \Delta k_0 \pmod{m}.$$

Since by (1.2)

$$\Delta k_0 = k_m - 2^m k_0 \equiv 2^m - 2^m k_0 \equiv 0 \pmod{m},$$

it follows at once that $\Delta k_n \equiv 0 \pmod{m}$. This proves the first half of (1.4).

In the next place, using (1.1) we get

$$(2.2) \quad K_{n+m} \equiv n^2 K_{n+m-1} + (n)_2 K_{n+m-2} + 2(n)_3 K_{n+m-3} + k_{n+m},$$

so that

$$(2.3) \quad \Delta K_n \equiv n^2 \Delta K_{n-1} + (n)_2 \Delta K_{n-2} + 2(n)_3 \Delta K_{n-3}.$$

Since by (1.1)

$$\begin{aligned} K_m &\equiv k_m, & K_{m+1} &\equiv K_m + k_{m+1} \equiv k_m + k_{m+1} \equiv 0, \\ K_{m+2} &\equiv 4K_{m+1} + 2K_m + k_{m+2} \equiv 2k_m + k_{m+2} \equiv 0 \end{aligned}$$

(using the special values $k_0=1, k_1=-1, k_2=-2, K_0=1, K_1=K_2=0$), it follows that

$$(2.4) \quad \Delta K_0 \equiv \Delta K_1 \equiv \Delta K_2 \equiv 0 \pmod{m}.$$

Clearly (2.3) and (2.4) imply

$$(2.5) \quad \Delta K_n \equiv 0 \pmod{m}$$

for all $n \geq 0$. This completes the proof of (1.4).

3. Proof of (1.6). We shall require an extension of (2.1). Replacing n by $n+m$, we get

$$\Delta^2 k_n + n\Delta^2 k_{n-1} + 2m\Delta k_{n+m-1} = 0,$$

and it is then easy to get the general formula

$$(3.1) \quad \Delta^r k_n + n\Delta^r k_{n-1} + rm\Delta^{r-1} k_{n+m-1} = 0 \quad (n \geq 0)$$

for $r \geq 2$. We now use (3.1) to prove

$$(3.2) \quad \Delta^r k_n \equiv 0 \pmod{m^r} \quad (r \geq 1).$$

Indeed we have already proved (3.2) for the value $r=1$. If then we assume (3.2) for the value $r-1$, (3.1) implies

$$(3.3) \quad \Delta^r k_n \equiv -n\Delta^r k_{n-1} \equiv (-1)^n n! \Delta^r k_0.$$

Now if we take $n=0$ in (3.1) we get

$$\Delta^r k_0 \equiv -rm\Delta^{r-1}k_{m-1} \equiv 0,$$

by the inductive hypothesis. Hence (3.3) reduces to (3.2). This proves the first half of (1.6).

We now prove

$$(3.4) \quad \Delta^r K_n \equiv 0 \pmod{m^r} \quad (r \geq 1).$$

By (2.5), (3.4) holds for $r=1$; we therefore assume that it holds for the value $r-1$. Now it follows from (1.1) that

$$(3.5) \quad \begin{aligned} \Delta^r K_n &= n^2 \Delta^r K_{n-1} + (n)_2 \Delta^r K_{n-2} + 2(n)_3 \Delta^r K_{n-3} + \Delta^r k_n \\ &+ r \{ (\Delta n^2) \Delta^{r-1} K_{n+m-1} + (\Delta(n)_2) \Delta^{r-1} K_{n+m-2} \\ &\quad + 2(\Delta(n)_3) \Delta^{r-1} K_{n+m-3} \} + \dots \end{aligned}$$

for all $n \geq 0$. Since $\Delta^r n^k \equiv 0 \pmod{m^r}$ for $k \geq 0$, it is evident from (3.5) that we need merely examine $\Delta^r K_n$ for $n=0, 1, 2$. In the first place (3.5) implies $\Delta^r K_0 \equiv 0 \pmod{m^r}$ by the inductive hypothesis. Secondly for $n=1$, we see that $\Delta^r K_1 \equiv \Delta^r K_0 \equiv 0$, and for $n=2$, $\Delta^r K_2 \equiv 4\Delta^r K_1 + 2\Delta^r K_0 \equiv 0$. Thus $\Delta^r K_n \equiv 0$ for all $n \geq 0$. This completes the proof of (1.6).

REFERENCE

1. John Riordan, *A recurrence relation for three-line latin rectangles*, Amer. Math. Monthly vol. 59 (1952) pp. 159-162.

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