

ON CONTINUITY PROPERTIES OF DERIVATIVES OF SEQUENCES OF FUNCTIONS

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It is the purpose of this note to indicate some readily proved results concerning the p th derivatives of convergent sequences of functions of a real variable; these results are associated with repeated term-by-term differentiation, and involve especially values assumed, total variation, and modulus of continuity of p th derivatives.

As an illustration of this material, we remark that it is intuitively obvious that if a sequence of functions $f_n(x)$ possessing derivatives approaches the function $\cos x$ in the interval $-1 \leq x \leq +1$, not necessarily uniformly, then for n sufficiently large the function $f'_n(x)$ has at least one zero for some x near $x=0$; the same conclusion holds if the sequence $f_n(x)$ approaches the function $|x|$ in the interval $-1 \leq x \leq +1$.

Although the two investigations were undertaken independently, the present note has close connections with a forthcoming paper by Ulam and Hyers. The latter authors emphasize consequences of *uniform* convergence of a sequence, but under appropriate circumstances study the values taken on by, and especially the vanishing of, the p th derivatives of the functions of an approximating sequence; they also investigate analogous problems for functions of several variables.

THEOREM 1. *Let the functions $f_n(x)$ converge to the function $f(x)$ in the interval $I: a \leq x \leq b$, and let both $f_n(x)$ and $f(x)$ possess derivatives of order p (>0) at every point of I . Let there be given a point x_0 of I , and positive numbers δ and ϵ . Then there exists N such that for every $n > N$ the function $f_n^{(p)}(x)$ takes on a value $f_n^{(p)}(x_n)$ which satisfies*

$$(1) \quad |f_n^{(p)}(x_n) - f^{(p)}(x_0)| < \epsilon$$

at some point x_n of the interval $|x - x_0| < \delta$.

Consider first the case $p=1$. At a suitably chosen point x'_0 of I near x_0 we have

$$(2) \quad \left| \frac{f(x'_0) - f(x_0)}{x'_0 - x_0} - f'(x_0) \right| < \frac{\epsilon}{2}, \quad |x_0 - x'_0| < \delta.$$

For n sufficiently large we have by the convergence of the sequence

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$f_n(x)$ in the points x'_0 and x_0

$$(3) \quad \left| \frac{f_n(x'_0) - f_n(x_0)}{x'_0 - x_0} - \frac{f(x'_0) - f(x_0)}{x'_0 - x_0} \right| < \frac{\epsilon}{2}.$$

But the first fraction in (3) has the value $f'_n(x_n)$, where x_n is a suitably chosen point in the interval $|x - x_0| < \delta$, so inequality (1) for $p=1$ follows from (2) and (3).

It is now clear how the proof of Theorem 1 can be completed by induction. Assume the theorem true for the index $p-1$; we prove the theorem for the index p . We chose x'_0 in I satisfying

$$(4) \quad \left| \frac{f^{(p-1)}(x'_0) - f^{(p-1)}(x_0)}{x'_0 - x_0} - f^{(p)}(x_0) \right| < \epsilon, \quad |x_0 - x'_0| < \delta.$$

The function $f^{(p-1)}$ possesses a derivative and hence is continuous in I , so the corresponding inequality is valid if in the denominator of the fraction the values x'_0 and x_0 are replaced by arbitrary values X'_0 and X_0 in suitable neighborhoods $N(x'_0)$ and $N(x_0)$ of x'_0 and x_0 respectively (these neighborhoods are to be chosen to lie in $|x - x_0| < \delta$), and if in the numerator of the fraction the values $f^{(p-1)}(x'_0)$ and $f^{(p-1)}(x_0)$ are replaced by arbitrary values g' and g satisfying suitable inequalities

$$(5) \quad |f^{(p-1)}(x'_0) - g'| < \epsilon_1, \quad |f^{(p-1)}(x_0) - g| < \epsilon_1.$$

That is to say, if X'_0 and X_0 lie in $N(x'_0)$ and $N(x_0)$, and if (5) is valid, then we have

$$(6) \quad \left| \frac{g' - g}{X'_0 - X_0} - f^{(p)}(x_0) \right| < \epsilon.$$

By Theorem 1 as assumed true for the index $p-1$, there exists N so that for $n > N$ the function $f_n^{(p-1)}(x)$ takes on a value g' satisfying (5) in some point X'_0 of $N(x'_0)$ and simultaneously takes on a value g satisfying (5) in some point X_0 of $N(x_0)$; here X'_0 and X_0 naturally depend on n . For such values of n we have

$$\left| \frac{f_n^{(p-1)}(X'_0) - f_n^{(p-1)}(X_0)}{X'_0 - X_0} - f^{(p)}(x_0) \right| < \epsilon.$$

The fraction is equal to $f_n^{(p)}(x_n)$ in some point x_n between X'_0 and X_0 , so x_n lies in the interval $|x - x_0| < \delta$, and Theorem 1 is established.

We remark that at the end points of I we deal wholly with one-sided derivatives of $f(x)$ and $f_n(x)$; it follows that the prescribed interval for x_n may be restricted to a one-sided neighborhood also if

x_0 is an interior point of I .

Ulam and Hyers consider Theorem 1 in the case $f^{(p)}(x_0) = 0$, where $f^{(p)}(x)$ changes sign at $x = x_0$, and require uniform convergence of the sequence $f_n(x)$; their method involves the use of m th differences, and can be combined with the present methods to establish Theorem 1.

Proof of the following is essentially contained in the discussion as given:

COROLLARY 1. *Under the hypothesis of Theorem 1, let x_1 and x_2 ($< x_1$) lie in I ; then for n sufficiently large there exist X_1 and X_2 in I depending on n such that*

$$\left| \frac{f_n^{(p)}(X_1) - f_n^{(p)}(X_2)}{X_1 - X_2} - \frac{f^{(p)}(x_1) - f^{(p)}(x_2)}{x_1 - x_2} \right| < \epsilon$$

for some points X_1 and X_2 with $|X_1 - x_1| < \delta$, $|X_2 - x_2| < \delta$. In particular if $f_n^{(p+1)}(x)$ exists at every point of I , for n sufficiently large there exists some point X , $x_2 < X < x_1$, such that we have

$$\left| f_n^{(p+1)}(X) - \frac{f^{(p)}(x_1) - f^{(p)}(x_2)}{x_1 - x_2} \right| < \epsilon.$$

If x_1 and x_2 ($< x_1$) are arbitrary points of I , and if we have $f^{(p)}(x_1) > f^{(p)}(x_2)$ [or $< f^{(p)}(x_2)$], and if $f^{(p)}(x_1) > A > f^{(p)}(x_2)$ [or $f^{(p)}(x_1) < A < f^{(p)}(x_2)$], then for n sufficiently large $f_n^{(p)}(x)$ takes on the value A in some point X_n , $x_1 < X_n < x_2$.

The last remark follows from Theorem 1 and the classical property of the derivative $f_n^{(p)}(x)$.

Both Theorem 1 and Corollary 1 are of significance in the study of approach by functions $f_n(x)$ having more derivatives than the limit function $f(x)$. The interval I of Theorem 1 may be only a subinterval of a larger interval of convergence. For instance suppose $f_n(x) \rightarrow f(x) \equiv |x|$ in I' : $-1 \leq x \leq 1$. Suppose $f_n'(x)$ exists at every point of I' , and let $\delta (> 0)$ be given. For n sufficiently large, it follows from Theorem 1 that $f_n'(x)$ takes a value near unity in a neighborhood of the point $\delta/2$ and takes a value near minus unity in a neighborhood of the point $-\delta/2$, hence that $f_n'(x)$ takes the value zero in some point of the interval $|x| \leq \delta$; compare the second part of Corollary 1. If $f_n''(x)$ exists and is continuous in I' , the equation

$$f_n'(x_1) - f_n'(x_2) = \int_{x_2}^{x_1} f_n''(x) dx,$$

where x_1 and x_2 are near $\delta/2$ and $-\delta/2$ respectively, shows that for

n sufficiently large, $f_n''(x)$ takes some value numerically greater than $1/\delta$ at some point of the interval $|x| < \delta$. Extension of this reasoning shows that if $f_n^{(p)}(x)$ exists and is continuous at every point of I' , and if M and δ are arbitrary, then for n sufficiently large $f_n^{(p)}(x)$ takes some value numerically greater than M in the interval $|x| < \delta$. Of course Corollary 1 extends to higher difference quotients.

These remarks concerning approximation to the function $|x|$ are closely related to the more obvious fact that if $f(x)$ is defined throughout the interval I but is discontinuous at the point x_0 of I , and if the sequence of functions $f_n(x)$ continuous in I converges in I to $f(x)$, then if M and $\delta(>0)$ are given, for n sufficiently large the difference quotient of $f_n(x)$ is numerically greater than M at some point of the interval $|x - x_0| < \delta$; if $f_n'(x)$ exists throughout I , then for n sufficiently large $f_n'(x)$ is numerically greater than M at some point of the interval $|x - x_0| < \delta$; a similar conclusion applies to the higher derivatives of $f_n'(x)$ if they exist, for we cannot have here $f_n'(x) \rightarrow +\infty$ or $f_n'(x) \rightarrow -\infty$ in an interval, as is shown in Lemma 1 below. It follows similarly that if $f(x)$ is continuous in I but has no derivative at the point x_0 of I , if the sequence of functions $f_n(x)$ continuous in I converges in I to $f(x)$ and if $f_n'(x)$ exists in I , then if M and $\delta(>0)$ are given, for n sufficiently large the difference quotient of $f_n'(x)$ is numerically greater than M at some point of the interval $|x - x_0| < \delta$; if $f_n''(x)$ exists throughout I , then for n sufficiently large the second derivative $f_n''(x)$ is numerically greater than M at some point of the interval $|x - x_0| < \delta$; a similar conclusion applies to higher derivatives if they exist.

COROLLARY 2. *Under the conditions of Theorem 1 we have*

$$\liminf_{n \rightarrow \infty} [\text{Total variation of } f_n^{(p)}(x) \text{ in } I] \\ \geq [\text{Total variation of } f^{(p)}(x) \text{ in } I].$$

Corollary 2 is a direct consequence of Theorem 1 and the definition of the total variation of $f^{(p)}(x)$ in I as

$$\text{l.u.b.} \sum_{k=0}^K |f^{(p)}(\xi_{k+1}) - f^{(p)}(\xi_k)|, \quad a = \xi_0 < \xi_1 < \cdots < \xi_K < \xi_{K+1} = b;$$

the proof is left to the reader.

COROLLARY 3. *Under the conditions of Theorem 1, if $\omega_n(\delta)$ is a modulus of continuity in I for the function $f_n^{(p)}(x)$ (assumed continuous), and $\omega(\delta)$ is the least modulus of continuity in I for $f^{(p)}(x)$ (assumed continuous), then we have for every δ*

$$\liminf_{n \rightarrow \infty} \omega_n(\delta) \geq \omega(\delta).$$

The function $\omega(\delta)$ is said to be a *modulus of continuity* for the continuous function $\psi(x)$ in I if we have $|\psi(x+h) - \psi(x)| \leq \omega(\delta)$ whenever x and $x+h$ lie in I , with $|h| \leq \delta$, and if $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$. Under the hypothesis of Corollary 3 let δ and $\epsilon (> 0)$ be arbitrary. There exist x and $x+h$ in I satisfying $|f^{(p)}(x+h) - f^{(p)}(x)| \geq \omega(\delta) - \epsilon/3$, $|h| < \delta$. Choose $\delta_1 (> 0)$, $|h| + 2\delta_1 \leq \delta$. For n sufficiently large we have for some x_1 and x_2 (depending on n)

$$\begin{aligned} |f_n^{(p)}(x_1) - f^{(p)}(x)| &< \epsilon/3, & |x - x_1| &< \delta_1, \\ |f_n^{(p)}(x_2) - f^{(p)}(x+h)| &< \epsilon/3, & |x+h - x_2| &< \delta_1, \end{aligned}$$

from which we may write (since $|x_1 - x_2| < \delta$)

$$\omega_n(\delta) \geq |f_n^{(p)}(x_1) - f_n^{(p)}(x_2)| \geq \omega(\delta) - \epsilon,$$

whence the conclusion follows.

In this same circle of ideas we prove

THEOREM 2. *Suppose all the functions $f_n^{(p)}(x)$ are continuous and have the modulus of continuity $\omega(\delta)$ in the interval $I: a \leq x \leq b$, and suppose the sequence $f_n(x)$ converges to a function $f(x)$ on I . Then $f^{(p)}(x)$ exists on I and has there the modulus of continuity $\omega(\delta)$.*

The functions $f_n(x)$ are in fact equicontinuous on I , so it is sufficient to suppose the sequence $f_n(x)$ convergent to $f(x)$ on a set everywhere dense in I ; indeed it is sufficient for the existence of $f(x)$ to assume the sequence $f_n(x)$ convergent in $p+1$ points of I . To prove Theorem 2 in the case $p > 1$ we need two lemmas.

LEMMA 1. *If $f'_n(x)$ is continuous in I and becomes positively infinite there uniformly, then there is at most one point in I at which $f_n(x)$ converges or is bounded. If such a point ξ_0 exists, in any closed subinterval of I to the right of ξ_0 we have uniformly $f_n(x) \rightarrow +\infty$, and in any closed subinterval of I to the left of ξ_0 we have uniformly $f_n(x) \rightarrow -\infty$.*

If there exist two points ξ_0 and $\xi_1 (> \xi_0)$ in I at which $f_n(x)$ converges (or is bounded), we have

$$f_n(\xi_1) - f_n(\xi_0) = \int_{\xi_0}^{\xi_1} f'_n(x) dx \rightarrow +\infty,$$

which is impossible.

This last equation also shows that if $f_n(\xi_0)$ converges, then $f_n(\xi_1) \rightarrow +\infty$ for every ξ_1 in I with $\xi_1 > \xi_0$, and a similar equation shows that

if $f_n(\xi_1) \rightarrow +\infty$ then $f_n(x) \rightarrow +\infty$ uniformly in I for $x \geq \xi_1$. It may be proved similarly that if $f_n(\xi_0)$ converges, then at every x in I to the left of ξ_0 we have $f_n(x) \rightarrow -\infty$, uniformly on any closed subinterval of I to the left of ξ_0 .

LEMMA 2. *If $f_n^{(p)}(x)$ is continuous in I and becomes positively infinite there uniformly, there are at most p points of I at which the sequence $f_n(x)$ converges or is bounded. If there are p such points of I , these points divide I into at most $p+1$ subintervals I_j ; interior to each I_j we have $f_n(x) \rightarrow +\infty$ or $f_n(x) \rightarrow -\infty$, uniformly on any closed subinterval interior to I_j .*

Lemma 2 is a consequence of Lemma 1, by application of Lemma 1 to the subintervals of I in which $f_n^{(p-1)}(x)$ becomes positively and negatively infinite, respectively. Under the hypothesis of Lemma 2 with $p=2$, suppose the sequence $f'_n(x)$ convergent or even bounded in a point ξ_0 interior to I ; it cannot occur that $f_n(x)$ should converge at ξ_0 as well as at a point $\xi_1 (> \xi_0)$ of I and at a point $\xi_2 (< \xi_0)$ of I , for under those conditions by Lemma 1 we should have $f_n(x) \rightarrow +\infty$ or $f_n(x) \rightarrow -\infty$ uniformly in a subinterval of each of the intervals $\xi_2 < x < \xi_0$, $\xi_0 < x < \xi_1$; each of the latter intervals contains for n sufficiently large a maximum or minimum of $f_n(x)$ and thus at least one zero of $f'_n(x)$, so for n sufficiently large I contains at least one zero of $f''_n(x)$, in contradiction to our hypothesis. Indeed it follows from this same reasoning applied to a suitable subsequence that ($p=2$) if the sequence $f_n(x)$ converges in each of two points η_1 and $\eta_2 (< \eta_1)$ of I , then in each of the subintervals of I that exist: $a \leq x < \eta_2$, $\eta_2 < x < \eta_1$, $\eta_1 < x \leq b$ we have $f_n(x) \rightarrow +\infty$ or $f_n(x) \rightarrow -\infty$ and uniformly in each closed subsubinterval. Continued application of this argument establishes Lemma 2.

The number $p+1$ of subintervals I_j of Lemma 2 may actually be attained, as is shown by the example

$${}_n(x) \equiv n \left(x - \frac{1}{p+1} \right) \left(x - \frac{2}{p+1} \right) \cdots \left(x - \frac{p}{p+1} \right),$$

$$I: 0 \leq x \leq 1,$$

a polynomial of degree p , whose p th derivative becomes positively infinite. The relations $f_n(x) \rightarrow +\infty$ and $f_n(x) \rightarrow -\infty$ of Lemma 2 do not necessarily hold in alternate intervals I_j , as we see from the example $f_n(x) \equiv n(x-1/2)^2$, $I: 0 \leq x \leq 1$, with $p=2$; we have $f_n^{(p)}(x) \equiv 2n \rightarrow +\infty$.

We return to the proof of Theorem 2. The functions $f_n^{(p)}(x)$ are

equicontinuous on I , so every subsequence which is bounded in a point of I is uniformly bounded in I ; any subsequence which becomes positively (or negatively) infinite in a point of I becomes positively (or negatively) infinite uniformly in I . From Lemma 2 it follows that no subsequence can become positively or negatively infinite uniformly in I , so the functions $f_n^{(p)}(x)$ are uniformly bounded in I . The functions $f_n^{(p-1)}(x)$ have their first derivatives uniformly bounded in I , hence are equicontinuous in I . By the argument just given for the functions $f_n^{(p)}(x)$, and by application of Lemma 2, it follows that the set $f_n^{(p-1)}(x)$ is uniformly bounded in I , and by continuing this argument it follows that each of the sets $f_n^{(p-2)}(x), \dots, f_n'(x), f_n(x)$ is uniformly bounded and equicontinuous in I . Then for a suitably chosen sequence of integers n_k , it is true that at a set of points everywhere dense in I , each of the sequences $f_{n_k}(x), f_{n_k}'(x), \dots, f_{n_k}^{(p)}(x)$ converges, hence converges uniformly in I to some limit function; we denote these limit functions by $F_0(x), F_1(x), \dots, F_p(x)$. From the hypothesis of Theorem 2 we have $F_0(x) \equiv f(x)$. From the uniformity of the convergence of the sequence $f_{n_k}'(x)$ it follows by the classical theorem on term-by-term differentiation of series that $f'(x)$ exists and we have $F_1(x) \equiv f'(x)$. Repetition of this reasoning shows that $f''(x), \dots, f^{(p)}(x)$ all exist and we have $F_2(x) \equiv f''(x), \dots, F_p(x) \equiv f^{(p)}(x)$. The remainder of Theorem 2 follows from Corollary 3 to Theorem 1.

Under the hypothesis of Theorem 2, we have essentially shown that every subsequence of the functions $f_n^{(p)}(x)$ contains a new subsequence which converges uniformly in I to $f^{(p)}(x)$, from which it follows that the sequence $f_n^{(p)}(x)$ itself converges uniformly in I to $f^{(p)}(x)$, and that the sequences $\{f_n^{(p-1)}(x)\}, \{f_n^{(p-2)}(x)\}, \dots, \{f_n(x)\}$ converge uniformly in I to the respective limits $f^{(p-1)}(x), f^{(p-2)}(x), \dots, f(x)$.

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