DETERMINATION OF THE TYPE AND PROPERTIES OF THE MAPPING FUNCTION OF A CLASS OF DOUBLY-CONNECTED RIEMANN SURFACES

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1. **Introduction.**[1] The object of this paper is to consider a class of open doubly-connected Riemann surfaces[2] and to show that the members of the class are images of the complex plane less two points; also a representation of the mapping function is obtained. To this end a class of open simply-connected Riemann surfaces is defined and the members shown to be parabolic and a representation of the mapping function is obtained.

The methods employed involve approximation by sequences of elliptic surfaces developed by G. R. MacLane [8][3] and the use of the results of Carathéodory [2] on the mapping of a sequence of plane domains by a family of functions.

In the terminology of Iversen [7] certain members of each class of surfaces exhibit indirectly critical singularities.

2. **Description of the class of simply-connected surfaces.** A surface $\mathcal{J}$ of this class is defined by two infinite sequences of real numbers $\{a_n\}$, $\{b_n\}$ $(n=1, 2, \cdots)$ with $0<a_1<b_1$, $a_{2k\pm 1}>a_{2k}$, $b_{2k-1}>a_{2k-1}$, $a_{2k}>b_{2k}$. $\mathcal{J}$ consists of sheets $S_1, S_2, \cdots, S_k, \cdots$, each sheet being a slit copy of the w-sphere. S_1 is cut along the positive real axis from $w=a_1$ to $w=b_1$. S_k $(k>1)$ is cut along the real axis from a_{k-1} to b_{k-1} and from a_k to b_k. S_1 and S_2 are joined along their cuts from a_1 to b_1 forming first order branch points over a_1 and b_1. S_{2k} and S_{2k+1} are joined along their cuts from a_{2k} to b_{2k}, S_{2k} and S_{2k-1} are joined along their cuts from a_{2k-1} to b_{2k-1} forming first order branch points over $w=a_n$ and $w=b_n$, $n=1, 2, 3, \cdots$. $\mathcal{J}$ is topographically equivalent to a semi-infinite cylinder, hence is open and simply-connected. Therefore $\mathcal{J}$ is either hyperbolic or parabolic.

The nature of the singularities of $\mathcal{J}$ depends on the sequences $\{a_n\}$, $\{b_n\}$. (1) If neither sequence has a limit, $\mathcal{J}$ has no singularities. (2) If one of the sequences has a limit or if both have the same limit,

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[3] Numbers in brackets refer to the references listed at the end of the paper.

$\mathcal{J}$ has one indirectly critical singularity. (3) If $a_n \to a \neq \infty$ and $b_n \to \infty$, $\mathcal{J}$ has two indirectly critical singularities.

3. **Description of the class of doubly-connected surfaces.** A surface $\mathcal{J}$ of this class is defined by two infinite sequences of real numbers $\{a_n\}$, $\{b_n\}$, $n = \pm 1, \pm 2, \pm 3, \cdots$, with $b_{-1} < a_{-1} < 0 < a_1 < b_1$, $a_{2k\pm 1} > a_{2k}$, $b_{2k-1} > a_{2k-1}$, $a_{2k} > b_{2k}$, $a_{-2k} > a_{-2k\pm 1}$, $a_{-2k-1} > b_{-2k-1}$, $b_{-2k} > a_{-2k}$, $k \geqq 1$. $\mathcal{J}$ consists of an infinite number of sheets

$$\cdots, S_{-k}, S_{-k+1}, \cdots, S_{-1}, S_1, S_2, \cdots, S_k, S_{k+1}, \cdots, \qquad k = 1, 2, 3, \cdots,$$

each sheet being a slit copy of the w-sphere. $S_j, j = -1, \pm 2, \pm 3, \cdots$, is cut along the real axis from a_j to b_j and from a_{j-1} to b_{j-1}. S_1 is cut from a_1 to b_1 and from a_{-1} to b_{-1}. S_j and S_{j+1} are joined along their cuts between a_j and b_j for $j \neq -1$. S_{-1} and S_1 are joined along their cuts between a_{-1} and b_{-1}. Branch points of first order are formed over $w = a_j$ and $w = b_j$, $j = \pm 1, \pm 2, \pm 3, \cdots$. $\mathcal{J}$ is topologically equivalent to an infinite cylinder, hence is open and doubly-connected. By the uniformizing principle [6], $\mathcal{J}$ can be mapped onto the ζ-plane slit along two line segments parallel to the real axis and thence onto the annulus $0 \leqq r < |z| < R \leqq \infty$ in the z-plane [3, pp. 71–72].

$\mathcal{J}$ can have no, one, two, three, or four singularities. For consider $\mathcal{J}$ as made up of two surfaces $\mathcal{J}_1$ and $\mathcal{J}_2$ where $\mathcal{J}_1$ consists of sheets of positive subscript and $\mathcal{J}_2$ of sheets of negative subscript, and the singularities of $\mathcal{J}_1$ and $\mathcal{J}_2$ can be classified as in §2. So any singularities of $\mathcal{J}$ will be indirectly critical.

4. **Proof that all surfaces of the first class are parabolic.** Let $\mathcal{J}$, a member of the class of simply-connected surfaces, be mapped onto the circle $|z| < R \leqq \infty$ by the function

$$(1) \qquad z = \phi(w), \quad w = f(z), \quad f(0) = 0 \in S_1, \quad f'(0) = 1.$$

To determine the images in the z-plane of the branch points of $\mathcal{J}$, consider the two symmetric halves of $\mathcal{J}$ obtained by slicing each sheet along the uncut portion of the real axis (i.e., S_k is sliced from b_k to $(-1)^{k+1}\infty$, from b_{k-1} to $(-1)^k\infty$, and from a_k to a_{k-1}). The half of $\mathcal{J}$ containing the upper half of S_1 can be mapped on a semi-circle $|z| < R \leqq \infty$, $\Im(z) > 0$, so that the point over the origin in S_1 is mapped onto $z = 0$, the point over ∞ in S_1 is mapped onto $z = \gamma_1 < 0$, the edges of the slices from $-\infty$ to a_1 and from a_{k-1} to a_k are mapped on the diameter $-R < z < R$ to the right of γ_1, and the edges of the slices from

b_k to $(-1)^{k+1}\infty$ are mapped on the diameter to the left of γ_1. This mapping carries the branch points over $\{a_n\}$ into a monotone increasing sequence of points $\{\alpha_n\}$ on the positive real axis and the branch points over $\{b_n\}$ into a monotone decreasing sequence of points $\{\beta_n\}$ on the negative real axis of the z-plane. The point over ∞ in S_k is carried into the point γ_k on the real axis with $\beta_k<\gamma_k<\beta_{k-1}$. If we apply the Schwartz reflection principle to the inverse of this mapping function and normalize, we are led to the function (1), so $f(z)$ is real for z real and the image of the branch point of $\mathcal{J}$ over a_k is $z=\alpha_k$ with $0<\alpha_1<\alpha_2< \cdots <\alpha_k<\alpha_{k+1}< \cdots$ and the image of the branch point of $\mathcal{J}$ over b_k is $z=\beta_k$ with $0>\beta_1>\beta_2> \cdots >\beta_k >\beta_{k+1}> \cdots$ and the image of the point of $\mathcal{J}$ in S_k over $w=\infty$ is $z=\gamma_k$ with $0>\gamma_1>\beta_1>\gamma_2>\beta_2> \cdots >\beta_{k-1}>\gamma_k>\beta_k> \cdots$. S_1 is mapped on a portion of $|z|<R$ bounded by a simple closed curve C_1, symmetric about the real axis and cutting the real axis in α_1 and β_1 only. S_k $(k>1)$ is mapped on a portion of $|z|<R$ bounded by two nonintersecting, simple, closed curves C_{k-1} and C_k, each symmetric about the real axis with C_{k-1} cutting the real axis at α_{k-1} and β_{k-1} only, and with C_k cutting the real axis at α_k and β_k only. The uncut segment of the real axis of S_1 corresponds to the segment (β_1, α_1) and the two shores of the cut from a_1 to b_1 correspond to the two symmetric halves of C_1. The uncut segment (a_{k-1}, a_k) of S_k $(k>1)$ corresponds to the segment (α_{k-1}, α_k) and the remaining uncut portion of the real axis of S_k corresponds to the segment (β_{k-1}, β_k). The shores of the cut (a_{k-1}, b_{k-1}) correspond to the two symmetric halves of C_{k-1} and the shores of the cut (a_k, b_k) correspond to the two symmetric halves of C_k. The curves C_k and the real axis are the paths on which $w=f(z)$ is real.

Consider the elliptic surface $\mathcal{J}_n$ which consists of the first $n+1$ sheets of $\mathcal{J}$ with the cut from a_{n+1} to b_{n+1} healed. $\mathcal{J}_n$ is a simply-connected closed surface with $2n$ first order branch points over $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ and with $n+1$ points over $w=\infty$. Hence $\mathcal{J}_n$ is the Riemann surface of the inverse of a rational function $w=R_n(z)$ which can be normalized so that $R_n(0)=0\in S_1$, $R_n'(0)=1$, $R_n(\infty)=\infty\in S_{n+1}$. $R_n(z)$ has $n+1$ simple poles, n of them at the images $\gamma_{1,n}, \gamma_{2,n}, \gamma_{3,n}, \cdots, \gamma_{n,n}$ of the points over $w=\infty$ in the sheets $S_1, \cdots, S_n$ and one at $z=\infty$, and $R_n'(z)$ must have $2n$ first order zeros at the images $\alpha_{1,n}, \alpha_{2,n}, \cdots, \alpha_{n,n}, \beta_{1,n}, \beta_{2,n}, \cdots, \beta_{n,n}$ of $a_1, \cdots, a_n, b_1, \cdots, b_n$ and no other zeros, where $\beta_{n,n}<\gamma_{n,n}<\beta_{n-1,n} < \cdots <\beta_{1,n}<\gamma_{1,n}<0<\alpha_{1,n}<\alpha_{2,n}< \cdots <\alpha_{n,n}$. Hence $R_n(z) =P_n(z)/Q_n(z)$ where P_n is a polynomial of degree $n+1$ and Q_n is a polynomial of degree n, and we can write $R_n(z)=\int_0^z R_n'(t)dt$,

$R_n'(z) = \prod_{k=1}^n (1-z/\alpha_{k,n})(1-z/\beta_{k,n})/(1-z/\gamma_{k,n})^2$ where the residues of $R_n'(z)$ at the poles are zero.

Let D_n be the z-plane cut from $\alpha_{n,n}$ to ∞ along the positive real axis. D_n is mapped by $w = R_n(z)$ onto $\mathcal{J}_n$ with the sheet S_{n+1} cut from a_n to $(-1)^n\infty$ along the real axis. But $\zeta = \phi(w)$ maps this cut surface one-to-one on the domain Δ_n of the ζ-plane bounded by the curve C_{n+1} and the segments $(\beta_{n+1}, \gamma_{n+1})$, (α_n, α_{n+1}) and containing $\zeta = 0$. So $\zeta = \phi[R_n(z)] = \psi_n(z)$ provides a schlicht mapping of D_n onto Δ_n with $\psi_n(0) = 0$ and $\psi_n'(0) = 1$. The following is an immediate corollary of Koebe's distortion theorem: Let $w = f(z)$ be holomorphic in the z-plane slit from $z = R > 0$ to $+\infty$ along the real axis and map this schlitzbereich on a plane domain Δ of the w-plane subject to the conditions $f(0) = 0$, $f'(0) = 1$. Then the distance from $w = 0$ to the boundary of Δ is greater than or equal to R. Applying this, we have that *the distance from* $\zeta = 0$ *to the curve* C_{n+1} *is greater than* $\alpha_{n,n}$.

For $0 < z < \alpha_{1,n}$, $\prod_{k=1}^n (1-z/\beta_{k,n})/(1-z/\gamma_{k,n}) < 1$, $\prod_{k=1}^n (1-z/\gamma_{k,n}) > 1$, $\prod_{k=1}^n (1-z/\alpha_{k,n}) > 0$, $0 < R_n'(z) < \prod_{k=1}^n (1-z/\alpha_{k,n})$, and if $1/\bar{\alpha}_n = (1/n)\sum_{k=1}^n 1/\alpha_{k,n}$, $R_n'(z) < [(1/n)\sum_{k=1}^n (1-z/\alpha_{k,n})]^n = (1-z/\bar{\alpha}_n)^n$. So $a_1 = \int_0^{\alpha_{1,n}} R_n'(z)dz < \int_0^{\alpha_{1,n}} (1-z/\bar{\alpha}_n)^n dz < \int_0^{\bar{\alpha}_n} (1-z/\bar{\alpha}_n)^n dz = \bar{\alpha}_n/(n+1)$, and since $\sum_{k=1}^n 1/\alpha_{k,n} < n/(n+1)a_1 < 1/a_1$ we have for any ν, $1 \leqq \nu \leqq n$, $\nu/\alpha_{\nu,n} < \sum_{k=1}^n 1/\alpha_{k,n} < 1/a_1$, or $\alpha_{\nu,n} > a_1\nu$ for $\nu = 1, 2, \cdots, n$; $n = 1, 2, \cdots$. Therefore the distance from the origin to C_{n+1} is greater than $a_1 n$ for all n, and $\mathcal{J}$ is parabolic.

5. Structure of the mapping function for surfaces of the first class. We have shown above that D_n converges to $|z| < \infty$ and Δ_n converges to $|\zeta| < \infty$, and using the form of Carathéodory's theorem [2, pp. 118–126] on families of schlicht mappings which is stated in Bieberbach [1, p. 13], we conclude: $\phi[R_n(z)] \to z$ uniformly in $|z| \leqq r$ for any finite r and $R_n(z) \to \phi^{-1}(z) = f(z)$ uniformly in $|z| \leqq r$. By Hurwitz' theorem [5, p. 249], $\lim_{n\to\infty} \alpha_{k,n} = \alpha_k$, $\lim_{n\to\infty} \beta_{k,n} = \beta_k$, $\lim_{n\to\infty} \gamma_{k,n} = \gamma_k$.

There exists a $\delta > 0$ and an integer n_0 such that for $0 \leqq |z| < \delta$ and $n > n_0$, $f'(z)$ and $R_n'(z)$ differ from zero and $\log R_n'(z) \to \log f'(z)$, where we take the determination of the logarithm which is zero for $z = 0$. Now

$$\log R_n'(z) = z\sum_{k=1}^n (2/\gamma_{k,n} - 1/\alpha_{k,n} - 1/\beta_{k,n})$$
$$+ (z^2/2)\sum_{k=1}^n (2/(\gamma_{k,n})^2 - 1/(\alpha_{k,n})^2 - 1/(\beta_{k,n})^2) + \cdots.$$

Hence $\lim_{n\to\infty} \left|\sum_{k=1}^n (2/\gamma_{k,n} - 1/\alpha_{k,n} - 1/\beta_{k,n})\right|$ exists and is finite.

Since $\beta_{k,n}<\gamma_{k,n}<0$, we have $-\infty<\lim_{n\to\infty}\sum_{k=1}^{n}(-2/|\gamma_{k,n}|-1/\alpha_{k,n}+1/|\beta_{k,n}|)<-\limsup_{n\to\infty}\sum_{k=1}^{n}(1/|\beta_{k,n}|+1/\alpha_{k,n})<0$. So

$$
\tag{2}
\begin{gathered}
\limsup_{n\to\infty}\sum_{k=1}^{n}1/\alpha_{k,n}<\infty,\qquad \limsup_{n\to\infty}\sum_{k=1}^{n}1/|\beta_{k,n}|<\infty,\\
\limsup_{n\to\infty}\sum_{k=1}^{n}1/|\gamma_{k,n}|<\infty,
\end{gathered}
$$

and hence

$$
\tag{3}
\sum_{k=1}^{\infty}1/\alpha_k,\qquad \sum_{k=1}^{\infty}1/|\beta_k|,\qquad \sum_{k=1}^{\infty}1/|\gamma_k|\qquad \text{all converge.}
$$

Because of (3) the product $\pi(z)=\prod_{k=1}^{\infty}(1-z/\alpha_k)(1-z/\beta_k)/(1-z/\gamma_k)^2$ converges, and for $|z|<\delta$, $\lim_{n\to\infty}\log R_n'(z)/\pi(z)=\log f'(z)/\pi(z)$.

$$
\begin{aligned}
\log R_n'(z)/\pi(z)=\sum_{m=1}^{\infty}(z^m/m)\Bigg\{&\left(\sum_{k=1}^{n}2/(\gamma_{k,n})^m-\sum_{k=1}^{\infty}2/(\gamma_k)^m\right)\\
&-\left(\sum_{k=1}^{n}1/(\alpha_{k,n})^m-\sum_{k=1}^{\infty}1/(\alpha_k)^m\right)\\
&-\left(\sum_{k=1}^{n}1/(\beta_{k,n})^m-\sum_{k=1}^{\infty}1/(\beta_k)^m\right)\Bigg\}.
\end{aligned}
$$

Since $\lim_{n\to\infty}\gamma_{k,n}=\gamma_k$, $\left|\limsup_{n\to\infty}\left(\sum_{k=1}^{n}1/(\gamma_{k,n})^m-\sum_{k=1}^{\infty}1/(\gamma_k)^m\right)\right|=\left|\limsup_{n\to\infty}\left(\sum_{k=n_0}^{n}1/(\gamma_{k,n})^m-\sum_{k=n_0}^{\infty}1/(\gamma_k)^m\right)\right|$, and using (2) and (3), we have $\left|\limsup_{n\to\infty}\left(\sum_{k=n_0}^{\infty}1/(\gamma_{k,n})^m-\sum_{k=n_0}^{\infty}1/(\gamma_k)^m\right)\right|\leqq 2\sum_{k=n_0}^{n}1/k^m$ for all n_0 sufficiently large. So

$$
\lim_{n\to\infty}\left(\sum_{k=1}^{n}1/(\gamma_{k,n})^m-\sum_{k=1}^{\infty}1/(\gamma_k)^m\right)=0,\qquad m\geqq 2.
$$

Similarly we have

$$
\lim_{n\to\infty}\left(\sum_{k=1}^{n}1/(\alpha_{k,n})^m-\sum_{k=1}^{\infty}1/(\alpha_k)^m\right)=0
$$

and

$$
\lim_{n\to\infty}\left(\sum_{k=1}^{n}1/(\beta_{k,n})^m-\sum_{k=1}^{\infty}1/(\beta_k)^m\right)=0\qquad \text{for } m\geqq 2.
$$

Therefore $\log f'(z)/\pi(z)=\delta z$, δ real and $f'(z)=e^{\delta z}\pi(z)$.

$\pi(z)$ is a canonical product of genus zero, and for

$$
0<\xi_1\leqq\arg z\leqq\eta_1<\pi,\qquad \pi<\xi_2\leqq\arg z\leqq\eta_2<2\pi,
$$

it can be proved [9, chap. VIII] that $|\pi(z)| = O[\exp(\epsilon|z|)]$ and $1/|\pi(z)| = O[\exp(\epsilon|z|)]$ for any $\epsilon > 0$. Hence

$$\exp(\delta\Re(z) - \epsilon|z|) \leqq |f'(z)| \leqq \exp(\delta\Re(z) + \epsilon|z|).$$

Suppose $\delta > 0$ and let $z \to \infty$ with $\pi/6 \leqq \arg z \leqq \pi/3$; we can find $\epsilon > 0$ such that $\exp(\delta\Re(z) - \epsilon|z|) \to \infty$. If $z \to \infty$ with $2\pi/3 \leqq \arg z \leqq 5\pi/6$, we can find $\epsilon > 0$ such that $\exp(\delta\Re(z) + \epsilon|z|) \to 0$. Now $f(z) = \int_0^z f'(t)dt$ so that there will exist a curve Γ_1 in $2\pi/3 \leqq \arg z \leqq 5\pi/6$ extending to ∞ along which $|f(z)| \to$ a finite value c, and there will exist a curve Γ_2 in $\pi/6 \leqq \arg z \leqq \pi/3$ extending to ∞ along which $|f(z)| \to \infty$. Consider the upper part of the path of reality C_{2j} and let z move along this path starting at α_{2j}. If $\delta > 0$, as z moves along C_{2j} for j sufficiently large $|f(z)|$ will attain an arbitrarily large value and then decrease to a finite value c_1 (arbitrarily near c). But the image $w = f(z)$ will move on the real axis starting at a_{2j} and moving *monotonically* to the left to b_{2j}, which is a contradiction and hence δ cannot be positive. If $\delta < 0$ we are not led to a contradiction, for in this case $|f(z)| \to c$ as z moves along the curve Γ_2 to ∞ and $|f(z)| \to \infty$ as z moves to ∞ along Γ_1. So that as z moves counterclockwise on C_{2j} or C_{2j+1}, $|f(z)|$ will increase from c_1 to an arbitrarily large value, and the image $w = f(z)$ will move either to the left on the real axis from a_{2k} or to the right on the real axis from a_{2k+1}. We have proved the following

THEOREM I. *Any Riemann surface belonging to the class described in* §2 *is parabolic and the mapping function is given by*

$$f(z) = \int_0^z f'(t)dt, \quad f'(z) = e^{\delta z}\prod_{k=1}^{\infty}(1 - z/\alpha_k)(1 - z/\beta_k)/(1 - z/\gamma_k)^2$$

where δ is a nonpositive real number, $\sum_{k=1}^{\infty} 1/\alpha_k < \infty$, $\sum_{k=1}^{\infty} 1/|\beta_k| < \infty$, $\sum_{k=1}^{\infty} 1/|\gamma_k| < \infty, \cdots, \beta_k < \gamma_k < \beta_{k-1} < \cdots < \beta_1 < \gamma_1 < 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < \alpha_{k+1} < \cdots$, *and the residues of $f'(z)$ at the poles γ_k are all zero.*

6. Proof that the members of the class of doubly-connected surfaces are images of the entire plane less two points. Let $\mathcal{J}$, a member of the class of doubly-connected surfaces, be mapped onto the annulus $0 \leqq r < |z| < R \leqq \infty$ by the function $z = \phi(w)$, $w = f(z)$. This mapping will be unique if we require $f(1) = 0 \in S_1$ and $f'(1) > 0$. By the methods mentioned in §4 the following can be obtained.

(1) $f(z)$ is real for z real.

(2) The image of the branch point over b_j is β_j, the image of the branch point over a_j is α_j, the image of the point of $\mathcal{J}$ on S_j over

$w=\infty$ is $z=\gamma_j$, $j=\pm 1, \pm 2, \pm 3, \cdots$, with $\cdots\beta_k<\gamma_k<\beta_{k-1}<\cdots<\beta_1<\gamma_1<\beta_{-1}<\gamma_{-1}<\cdots<\beta_{-k}<\gamma_{-k}<\beta_{-k-1}<\cdots<-r<0<r<\cdots<\alpha_{-k}<\alpha_{-k+1}<\cdots<\alpha_{-1}<1<\alpha_k<\alpha_{k+1}$, $k=1, 2, 3, \cdots$.

(3) S_1 is mapped on a portion of the annulus bounded by two nonintersecting, simple, closed curves C_1 and C_{-1}, symmetric about the real axis with C_1 cutting the real axis in α_1 and β_1 only, and C_{-1} cutting the real axis in α_{-1} and β_{-1} only. S_j, $j\neq 1$, is mapped on a portion of $0\leqq r<|z|<R\leqq\infty$ bounded by two nonintersecting, simple, closed curves C_j and C_{j-1}, symmetric about the real axis with C_j cutting the real axis in α_j and β_j only, and C_{j-1} cutting the real axis in α_{j-1} and β_{j-1} only.

That $r=0$ and $R=\infty$ follows from the parabolicity of the surfaces of the simply-connected class. Consider $\mathcal{J}$ as made up of two surfaces $\mathcal{J}_1$ and $\mathcal{J}_2$: $\mathcal{J}_1$ consisting of sheets $S_1, S_2, S_3, \cdots$, $\mathcal{J}_2$ consisting of sheets $S_{-1}, S_{-2}, S_{-3}, \cdots$, with $\mathcal{J}_1$ and $\mathcal{J}_2$ joined along the cut from a_{-1} to b_{-1}. Then $\mathcal{J}_1$ is mapped on a domain Δ_1 of the z-plane bounded by the curve C_{-1} and $|z|=R$. But by the results of §4, $\mathcal{J}_1$ can be mapped onto the plane $|\zeta|<\infty$ with the cut over (a_{-1}, b_{-1}) being mapped into a portion of the real axis (a, b). Hence the punched ζ-plane cut along the real axis from a to b can be mapped onto Δ_1 by a regular function with the cut (a, b) corresponding to C_{-1}. But then R must be infinite, for otherwise the circle $|z|=R<\infty$ would be the image of the point $\zeta=\infty$ by a regular function, which is impossible. By a similar argument we can show $r=0$. Therefore $\mathcal{J}$ can be mapped on the z-plane less the two points $z=0$, $z=\infty$.

7. Structure of the mapping function for surfaces of the second class. We have the portion of the z-plane exterior to the curve C_{-1} mapped one-to-one on $\mathcal{J}_1$ by $w=f(z)$. But from the results of §4 we have the ζ-plane less the point $\zeta=\infty$ and the segment C' of the real axis from $\phi(a_{-1})$ to $\phi(b_{-1})$ mapped one-to-one on $\mathcal{J}_1$ by $w=G_1(\zeta)$. Hence, the punched ζ-plane cut along C' can be mapped schlichtly by $z=h(\zeta)$ onto the portion of the z-plane exterior to C_{-1} in such a way that $\zeta=0$ corresponds to $z=1$, C' corresponds to C_{-1}, and $\zeta=\infty$ corresponds to $z=\infty$, i.e., $h(\zeta)$ must have a simple pole at $\zeta=\infty$. So $h(\zeta)=\zeta\psi(\zeta)$ where $\psi(\zeta)$ is regular at $\zeta=\infty$ and $\psi(\infty)\neq 0$, or $h(\zeta)=c\zeta+q(\zeta)$ where $q(\zeta)$ is regular at $\zeta=\infty$ and hence bounded in the neighborhood of $\zeta=\infty$. So we can write $h(\zeta)=\zeta+O(1)$ $(\zeta\to\infty)$. So $G_1(\zeta)=f[h(\zeta)]$, $G_1'(\zeta)=f'(z)h'(\zeta)$, $h'(\zeta)\neq 0$ for ζ sufficiently large. From the results of §5 we know that if the zeros of $G_1'(\zeta)$ are denoted by ζ_k, then $\sum_{k=1}^{\infty} 1/|\zeta_k|<\infty$. Since $z\sim c\zeta$ as $\zeta\to\infty$, the zeros α_k and β_k of $f'(z)$ are such that

$$\sum_{k=1}^{\infty} 1/|\alpha_k| < \infty, \qquad \sum_{k=1}^{\infty} 1/|\beta_k| < \infty. \tag{4}$$

In a similar manner we have the ζ-plane cut along a segment C'' of the real axis mapped one-to-one by $z=g(\zeta)$ onto the portion of the z-plane interior to C_1 in such a way that as $\zeta\to\infty$, $g(\zeta)\to 0$, i.e., $g(\zeta)$ must have a simple zero at $\zeta=\infty$. So $g(\zeta)=\psi(\zeta)/\zeta$ where $\psi(\infty)\neq 0$ and $\psi(\zeta)$ is regular at $\zeta=\infty$, or $g(\zeta)=c/\zeta+p(\zeta)$ where $p(\zeta)$ is $O(1/|\zeta|^2)$ as $\zeta\to\infty$. Then if $w=f(z)$ is the function mapping the interior of C_1 onto the surface $\mathcal{J}_2$ and $w=G_2(\zeta)$ is the function mapping the ζ-plane less the point $\zeta=\infty$ and the cut C'' onto $\mathcal{J}_2$, $G_2(\zeta)=f[g(\zeta)]$, $G_2'(\zeta)=f'(z)g'(\zeta)$, $g'(\zeta)\neq 0$ for ζ sufficiently large. Again from the results of §5 we know that if the zeros of $G_2'(\zeta)$ are ζ_k, then $\sum_{k=1}^{\infty} 1/|\zeta_k|<\infty$ so that, since $z\sim c/\zeta$ as $\zeta\to\infty$, the zeros of $f'(z)$, α_{-k}, β_{-k}, are such that $\sum_{k=1}^{\infty}|\alpha_{-k}|<\infty$, $\sum_{k=1}^{\infty}|\beta_{-k}|<\infty$.

Consider the surface $\mathcal{J}$ as the limit of a sequence of surfaces $\{\mathcal{J}_n\}$, where $\mathcal{J}_n$ consists of the $2n+1$ sheets $S_{-n}, S_{-n+1}, \cdots, S_{-1}, S_1, \cdots, S_{n+1}$ with the cuts (a_{-n-1}, b_{-n-1}) and (a_{n+1}, b_{n+1}) healed. $\mathcal{J}_n$ is the Riemann surface of the inverse of a rational function $w=R_n(z)$, $R_n(z)=P_n(z)/Q_n(z)$, $R_n(1)=0\in S_1$, $R_n'(1)>0$, $R_n(\infty)=\infty\in S_{n+1}$, $R_n(0)=\infty\in S_{-n}$, where $P_n(z)$ is a polynomial of degree $2n+1$ and $Q_n(z)$ is a polynomial of degree $2n$.

$$R_n'(z) = K_n \frac{\prod_{k=1}^{n}(1-z/\alpha_{k,n})(1-z/\beta_{k,n})(1-\alpha_{-k,n}/z)(1-\beta_{-k,n}/z)}{\prod_{k=1}^{n}(1-z/\gamma_{k,n})^2\prod_{k=1}^{n-1}(1-\gamma_{-k,n}/z)^2},$$

$K_n>0$, where $\alpha_{j,n}$, $\beta_{j,n}$ are the images of the points in the surface over a_j, b_j, and $\gamma_{j,n}$ is the image of the point over ∞ in sheet S_j, and the residues of $R_n'(z)$ at the poles are zero. $R_n(z)=\int_1^z R_n'(t)dt$. Now $\zeta=\phi[R_n(z)]$ maps the z-plane cut from $z=\alpha_{n,n}$ to ∞ and from $z=\alpha_{-n,n}$ to zero along the real axis schlichtly onto a domain bounded by C_{n+1}, C_{-n-1} cut along the segments

$$(\alpha_n, \alpha_{n+1}), \quad (\alpha_{-n-1}, \alpha_{-n}), \quad (\beta_{n+1}, \gamma_{n+1}), \quad \text{and} \quad (\gamma_{-n}, \beta_{-n-1}).$$

So using the fact that $r=0$ and $R=\infty$, and applying the theorem of Carathéodory mentioned in §5, we have

$$\lim_{n\to\infty} R_n(z) = f(z) \quad \text{and} \quad \lim_{n\to\infty} R_n'(z) = f'(z) \text{ uniformly.}$$

Hence $\lim_{n\to\infty}\alpha_{j,n}=\alpha_j$, $\lim_{n\to\infty}\beta_{j,n}=\beta_j$, $\lim_{n\to\infty}\gamma_{j,n}=\gamma_j$, $j=\pm 1, \pm 2, \pm 3, \cdots$, and we know

$$\sum_{k=1}^{\infty}\frac{1}{|\alpha_k|};\qquad \sum_{k=1}^{\infty}\frac{1}{|\beta_k|};\qquad \sum_{k=1}^{\infty}\frac{1}{|\gamma_k|};$$

$$\sum_{k=1}^{\infty}|\alpha_{-k}|;\qquad \sum_{k=1}^{\infty}|\beta_{-k}|;\qquad \sum_{k=1}^{\infty}|\gamma_{-k}|$$

all converge. For $|z-1|<\rho$ and $n>n_0$ we have $f'(z)\neq 0$ and $R_n'(z)\neq 0$, so for these values of z, $\lim_{n\to\infty}\log R_n'(z)=\log f'(z)$, taking the determination of the logarithm which is real for $z=1$. If we choose k_0 so that, for $|z-1|<\rho$ and $n>n_0$, $\max\,[|z/\beta_{k_0}|;\ |\beta_{-k_0,n}/z|]<1$ we can write

$$\begin{aligned}\log R_n'(z) = {} & \log\prod_{k=1}^{k_0}\frac{(1-z/\alpha_{k,n})(1-z/\beta_{k,n})(1-\alpha_{-k,n}/z)(1-\beta_{-k,n}/z)}{(1-z/\gamma_{k,n})^2(1-\gamma_{-k,n}/z)^2}\\
& + z\sum_{k=k_0}^{n}(2/\gamma_{k,n}-1/\alpha_{k,n}-1/\beta_{k,n})\\
& + \frac{z^2}{2}\sum_{k=k_0}^{n}[2/(\gamma_{k,n})^2-1/(\alpha_{k,n})^2-1/(\beta_{k,n})^2]+\cdots\\
& + \frac{1}{z}\sum_{k=k_0}^{n-1}(2\gamma_{-k,n}-\alpha_{-k,n}-\beta_{-k,n})\\
& + \frac{1}{2z^2}\sum_{k=k_0}^{n-1}[2(\gamma_{-k,n})^2-(\alpha_{-k,n})^2-(\beta_{-k,n})^2]+\cdots\\
& - \left[\frac{\alpha_{-n,n}}{z}+\frac{(\alpha_{-n,n})^2}{2z^2}+\cdots\right]\\
& - \left[\frac{\beta_{-n,n}}{z}+\frac{(\beta_{-n,n})^2}{2z^2}+\cdots\right]\\
& + \log K_n.\end{aligned}$$

Hence $\lim_{n\to\infty}\left|\sum_{k=k_0}^{n}(2/\gamma_{k,n}-1/\alpha_{k,n}-1/\beta_{k,n})\right|$ and

$$\lim_{n\to\infty}\left|\sum_{k=k_0+1}^{n}(2\gamma_{-k+1,n}-\alpha_{-k,n}-\beta_{-k,n})-\alpha_{k_0,n}-\beta_{k_0,n}\right|$$

exist and are finite. We have $0>\gamma_{k,n}>\beta_{k,n}$ or $-1/|\gamma_{k,n}|<-1/|\beta_{k,n}|$. Thus we have $-\infty<\lim_{n\to\infty}\sum_{k=k_0}^{n}(-2/|\gamma_{k,n}|-1/\alpha_{k,n}+1/|\beta_{k,n}|)$ $<\limsup_{n\to\infty}[-\sum_{k=k_0}^{n}(1/\alpha_{k,n}+1/|\beta_{k,n}|)]<0$. Therefore

$$\limsup_{n\to\infty}\sum_{k=k_0}^{n}1/|\beta_{k,n}|<\infty,\qquad \limsup_{n\to\infty}\sum_{k=k_0}^{n}1/\alpha_{k,n}<\infty,$$

$$\limsup_{n\to\infty} \sum_{k=k_0}^{n} 1/|\gamma_{k,n}| < \infty.$$

Also $0>\beta_{-k,n}>\gamma_{-k+1,n}$ or $-|\gamma_{-k+1,n}|<-|\beta_{-k,n}|$, hence

$$-\infty < \lim_{n\to\infty}\left[\sum_{k=k_0+1}^{n}(-2|\gamma_{-k+1,n}| - \alpha_{-k,n} + |\beta_{-k,n}|) - \alpha_{-k_0,n} + |\beta_{-k_0,n}|\right]$$

$$< \limsup_{n\to\infty} -\left[\sum_{k=k_0+1}^{n}(\alpha_{-k,n}+|\beta_{-k,n}|) + \alpha_{-k_0,n} - |\beta_{-k_0,n}|\right].$$

Therefore $\limsup_{n\to\infty}\sum_{k=k_0}^{n}\alpha_{-k,n}<\infty$, $\limsup_{n\to\infty}\sum_{k=k_0}^{n}|\beta_{-k,n}|<\infty$, $\limsup_{n\to\infty}\sum_{k=k_0}^{n-1}|\gamma_{-k,n}|<\infty$. Let

$$\pi(z) = \frac{\prod_{k=1}^{\infty}(1-z/\alpha_k)(1-z/\beta_k)(1-\alpha_{-k}/z)(1-\beta_{-k}/z)}{\prod_{k=1}^{\infty}(1-z/\gamma_k)^2(1-\gamma_{-k}/z)^2}.$$

Then

$$\begin{aligned}
\log\,[R_n'(z)/\pi(z)] &= \sum_{m=1}^{\infty}\left\{\frac{z^m}{m}\left[\left(\sum_{k=k_0}^{n}\frac{2}{(\gamma_{k,n})^m} - \sum_{k=k_0}^{\infty}\frac{2}{(\gamma_k)^m}\right)\right.\right.\\
&\quad - \left(\sum_{k=k_0}^{n}\frac{1}{(\alpha_{k,n})^m} - \sum_{k=k_0}^{\infty}\frac{1}{(\alpha_k)^m}\right)\\
&\quad \left.\left. - \left(\sum_{k=k_0}^{n}\frac{1}{(\beta_{k,n})^m} - \sum_{k=k_0}^{\infty}\frac{1}{(\beta_k)^m}\right)\right]\right\}\\
&\quad + \sum_{m=1}^{\infty}\left\{\frac{1}{mz^m}\left[\left(\sum_{k=k_0}^{n-1}2(\gamma_{-k,n})^m - \sum_{k=k_0}^{\infty}2(\gamma_{-k})^m\right)\right.\right.\\
&\quad - \left(\sum_{k=k_0}^{n-1}(\alpha_{-k,n})^m - \sum_{k=k_0}^{\infty}(\alpha_{-k})^m\right)\\
&\quad \left.\left. - \left(\sum_{k=k_0}^{n-1}(\beta_{-k,n})^m - \sum_{k=k_0}^{\infty}(\beta_{-k})^m\right)\right]\right\}\\
&\quad + \log K_n + T_n(z),
\end{aligned}$$

where $T_n(z)=o(1)$ as $n\to\infty$.

As in §5 it can be shown that, for $m\geqq 2$,

$$\lim_{n\to\infty}\left(\sum_{k=k_0}^{n} 1/(\gamma_{k,n})^m - \sum_{k=k_0}^{\infty} 1/(\gamma_k)^m\right),\quad \lim_{n\to\infty}\left(\sum_{k=k_0}^{n-1} (\gamma_{-k,n})^m - \sum_{k=k_0}^{\infty} (\gamma_{-k})^m\right),$$

$$\lim_{n\to\infty}\left(\sum_{k=k_0}^{n} 1/(\alpha_{k,n})^m - \sum_{k=k_0}^{\infty} 1/(\alpha_k)^m\right),\quad \lim_{n\to\infty}\left(\sum_{k=k_0}^{n} (\alpha_{-k,n})^m - \sum_{k=k_0}^{\infty} (\alpha_{-k})^m\right),$$

$$\lim_{n\to\infty}\left(\sum_{k=k_0}^{n} 1/(\beta_{k,n})^m - \sum_{k=k_0}^{\infty} 1/(\beta_k)^m\right),\quad \lim_{n\to\infty}\left(\sum_{k=k_0}^{n} (\beta_{-k,n})^m - \sum_{k=k_0}^{\infty} (\beta_{-k})^m\right)$$

all exist and are zero, and we have

$$\begin{aligned}
&\log f'(z)/\pi(z)\\
&\quad = \lim_{n\to\infty} \log R_n'(z)/\pi(z)\\
&\quad = z\lim_{n\to\infty}\left\{\left(\sum_{k=k_0}^{n} 2/\gamma_{k,n} - \sum_{k=k_0}^{\infty} 2/\gamma_k\right) - \left(\sum_{k=k_0}^{n} 1/\alpha_{k,n} - \sum_{k=k}^{\infty} 1/\alpha_k\right)\right.\\
&\qquad \left. - \left(\sum_{k=k_0}^{n} 1/\beta_{k,n} - \sum_{k=k_0}^{\infty} 1/\beta_k\right)\right\}\\
&\qquad + (1/z)\lim_{n\to\infty}\left\{\left(\sum_{k=k_0}^{n-1} 2\gamma_{-k,n} - \sum_{k=k_0}^{\infty} 2\gamma_{-k}\right)\right.\\
&\qquad \left. - \left(\sum_{k=k_0}^{n} \alpha_{-k,n} - \sum_{k=k_0}^{\infty} \alpha_{-k}\right) - \left(\sum_{k=k_0}^{n} \beta_{-k,n} - \sum_{k=k_0}^{\infty} \beta_{-k}\right)\right\}\\
&\qquad + \log K,
\end{aligned}$$

$\log f'(z)/\pi(z) = \sigma_1 z + \sigma_2/z + \log K$ where K and σ_1, σ_2 are real. So $f'(z) = K\,[\exp(\sigma_1 z + \sigma_2/z)]\pi(z)$.

With an argument similar to the one used in §5 it may be shown that σ_1 and σ_2 cannot be positive, so we have

THEOREM II. *Any Riemann surface belonging to the class described in* §3 *is the image of the z-plane less the two points* $z=\infty$ *and* $z=0$ *by a function* $w=f(z)$ *where*

$$f(z) = \int_1^z f'(t)dt,$$

$$f'(z) = K \exp(\sigma_1 z + \sigma_2/z)\prod_{k=1}^{\infty}\frac{(1-z/\alpha_k)(1-z/\beta_k)(1-\alpha_{-k}/z)(1-\beta_{-k}/z)}{(1-z/\gamma_k)^2(1-\gamma_{-k}/z)^2},$$

where $K>0$, σ_1, σ_2, α_j, β_j, γ_j $(j=1, \pm1, \pm2, \pm3, \cdots)$ *are real with* $\cdots \beta_k < \gamma_k < \beta_{+k-1} < \cdots < \beta_1 < \gamma_1 < \beta_{-1} < \gamma_{-1} < \cdots < \beta_{-k} < \gamma_{-k} < \cdots < 0 < \cdots < \alpha_{-k-1} < \alpha_{-k} < \cdots < \alpha_{-1} < \alpha_1 \cdots < \alpha_k \cdots,$

$$\sum_{k=1}^{\infty} 1/\alpha_k < \infty; \quad \sum_{k=1}^{\infty} 1/|\beta_k| < \infty; \quad \sum_{k=1}^{\infty} 1/|\gamma_k| < \infty;$$

$$\sum_{k=1}^{n} \alpha_{-k} < \infty; \quad \sum_{k=1}^{\infty} |\beta_{-k}| < \infty; \quad \sum_{k=1}^{\infty} |\gamma_{-k}| < \infty,$$

and the residues of $f'(z)$ at the poles are all zero, and σ_1, σ_2 are nonpositive.

8. Partial converse of Theorem I.

THEOREM III. *Let $w=f(z)$ be meromorphic in $|z|<\infty$ with $f(0)=0$, $f'(0)=1$. If $f(z)$ has the form $f(z)=ze^{\delta z}\prod_{k=1}^{\infty}(1-z/c_k)/(1-z/\gamma_k)$ where c_k, γ_k, δ are real, $\delta\leqq 0, \gamma_{k+1}<\gamma_k<0<c_k<c_{k+1}$ $(k=1, 2, \cdots)$ with $\sum_{k=1}^{\infty} 1/c_k<\infty$, $\sum_{k=1}^{\infty} 1/|\gamma_k|<\infty$, then the Riemann surface of the inverse of $f(z)$ is a surface of the class described in §2.*

Note that if in §2 the sequence $\{a_k\}$ is required to satisfy $a_{2k}<0<a_{2k\pm1}$, the function of Theorem I would satisfy the hypotheses of the above. The proof of Theorem III rests on the construction of a sequence of rational functions $\{R_n(z)\}$ which converges uniformly to $f(z)$ and whose members have derivatives of the same form as the $R_n'(z)$ in §2. The author has been unable to show that such a construction is possible without restricting the location of the zeros of $f(z)$ and so has been unable to give a proof of the complete converse of Theorem I although he conjectures its validity.

We distinguish two cases.

Case I: $\delta=0$. We may construct a sequence of rational functions $\{R_n(z)\}$ which converges uniformly to $f(z)$ by setting $R_n(z)=z\prod_{k=1}^{n}(1-z/c_k)/(1-z/\gamma_k)$. $R_n(z)$ has simple poles at $z=\gamma_k$, $k=1, 2, 3, \cdots, n$, and at $z=\infty$, and the residues at adjacent poles will differ in sign. Therefore there are at least n points $z=\beta_{k,n}$, $k=1, 2, 3, \cdots, n$, on the negative real axis such that $R_n'(\beta_{k,n})=0$ and the points will be distributed as follows: $-\infty<\beta_{n,n}<\gamma_n<\beta_{n-1,n}<\cdots<\gamma_{k+1}<\beta_{k,n}<\gamma_k<\cdots<\beta_{1,n}$. There will also be at least n points $z=\alpha_{k,n}$, $k=1, 2, 3, \cdots, n$, on the positive real axis such that $R_n'(\alpha_{k,n})=0$ and these points will be distributed as follows: $0<\alpha_{1,n}<c_1<\alpha_{2,n}<\cdots<c_{k-1}<\alpha_{k,n}<c_k<\cdots<\alpha_{n,n}<c_n$. Hence

$$R_n'(z) = \prod_{k=1}^{n}(1-z/\alpha_{k,n})(1-z/\beta_{k,n})/(1-z/\gamma_k)^2$$

and $R_n'(z)\to f'(z)$ uniformly.

It is obvious that the paths of reality consist of n simple, closed curves, symmetric about the real axis and intersecting the real axis at the points $z=\alpha_{k,n}$, $z=\beta_{k,n}$, $k=1, 2, 3, \cdots, n$. We assert that $\beta_{j,n}$, $\beta_{k,n}$, $k\neq j$, cannot lie on the same path of reality. For suppose that $\beta_{j,n}$, $\beta_{k,n}$ are both on the same path of reality C, then:

(1) If k, j are both odd or both even there will be an odd number of branch points between $\beta_{j,n}$, and $\beta_{k,n}$ and hence at least one path of reality would have to cross C at some point not on the real axis, which is impossible.

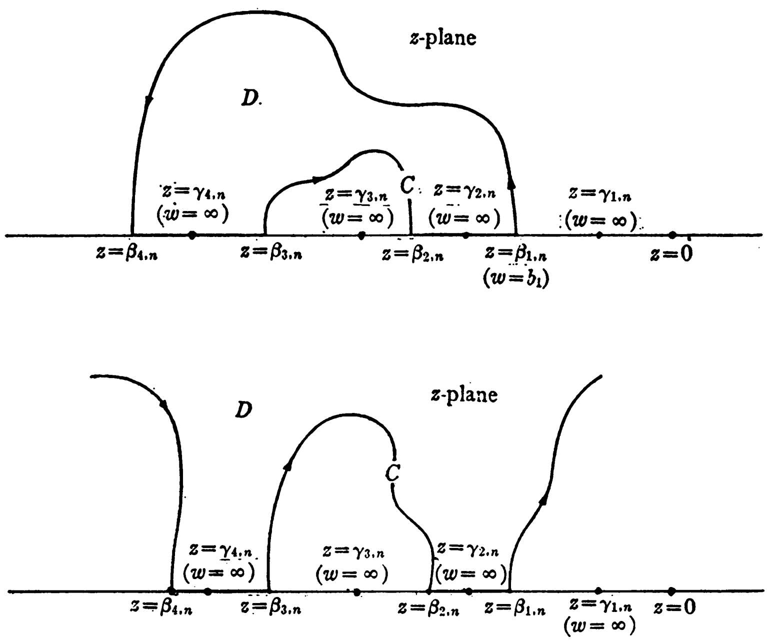

FIG. 1

(2) If k and j are not both odd or both even, as z traverses the path of reality around the domain D interior to the path, $R_n(z)$ would take on all real values twice (see Fig. 1), which is impossible. It is now apparent that the path of reality C_j through $\beta_{j,n}$ will intersect the real axis in the point $\alpha_{j,n}$, $j=1, 2, 3, \cdots, n$.

$R_n(z)=P_n(z)/Q_n(z)$ where $P_n(z)$ is a polynomial of degree $n+1$ and $Q_n(z)$ is a polynomial of degree n. Since $\overline{R_n(z)}=R_n(\bar{z})$ and $\overline{Q_n(z)}=Q_n(\bar{z})$ the equation of the paths of reality is given by

$$2i\Im[R_n(z)] = P_n(z)/Q_n(z) - P_n(\bar{z})/Q_n(\bar{z}) = 0$$

or

$$F(x, y) = P_n(x+iy)Q_n(x-iy) - P_n(x-iy)Q_n(x+iy) = 0.$$

From considerations of degree we see that any line $\arg z=\theta_0$, $\theta_0\neq 0, \pi$, will intersect the upper part of the paths of reality *only* once and the lower part of each of the paths of reality *only* once.

Collecting the foregoing we have: For any n, $R_n(z)$ is real for z on the real axis and for z on the noninteresecting, simple, closed curves $C_{1,n}, C_{2,n}, \cdots, C_{n,n}$ symmetric about the real axis with $C_{k,n}$ intersecting the real axis in the points $z=\alpha_{k,n}$, $z=\beta_{k,n}$. Furthermore, the curves $C_{k,n}$ are star-shaped (any ray *from the origin* intersects each curve only once).

If we denote by α_k and β_k the zeros of $f'(z)$, then $\alpha_{k,n}\to\alpha_k$ and $\beta_{k,n}\to\beta_k$ as $n\to\infty$, and $C_{k,n}$ will tend to a limiting curve C_k which is star-shaped, symmetric about the real axis, and passes through the points $z=\alpha_k$ and $z=\beta_k$. We wish to show that the curve C_k has no points at $z=\infty$. Consider the domain $D_{k,n}$ which contains $C_{k,n}$ and is bounded by $C_{k+1,n}$ and $C_{k-1,n}$. $D_{k,n}$ is mapped by $R_n(z)$ one-to-one onto a two-sheeted surface $\Sigma_{k,n}$ consisting of two slit copies of the w sphere S' and S''. S' is slit between $R_n(\alpha_{k-1,n})=a_{k-1,n}$ and $R_n(\beta_{k-1,n})=b_{k-1,n}$ and between $R_n(\alpha_{k,n})=a_{k,n}$ and $R_n(\beta_{k,n})=b_{k,n}$. S'' is slit between $a_{k,n}$ and $b_{k,n}$ and between $R_n(\alpha_{k+1,n})=a_{k+1,n}$ and $R_n(\beta_{k+1,n})=b_{k+1,n}$. The sheets are joined along their cuts from $a_{k,n}$ to $b_{k,n}$, and

$$R_n\left(\frac{\alpha_k+\alpha_{k+1}}{2}\right)=a\in S'',\qquad R_n'\left(\frac{\alpha_k+\alpha_{k+1}}{2}\right)=K.$$

$\Sigma_{k,n}$ can in turn be mapped by $\zeta=G(w)$ onto the domain $\Delta_{k,n}$ which is the ζ-plane slit from $G(a_{k-1,n})$ to $G(b_{k-1,n})$ and from $G(a_{k+1,n})$ to $G(b_{k+1,n})$. The segment $(a_{k,n}, b_{k,n})$ is mapped onto a closed curve $\Gamma_{k,n}$ and $G(a\in S'')=0$, $G'(a\in S'')=1$. The sequence of domains $\Delta_{k,n}$ converges to its kernel Δ_k which is the ζ-plane cut from $G(a_{k-1})$ to $G(b_{k-1})$ and from $G(a_{k+1})$ to $G(b_{k+1})$ where $a_j=\lim_{n\to\infty} R_n(\alpha_j)$, $b_j=\lim_{n\to\infty} R_n(\beta_j)$. Since the sequence of functions $\{G[R_n(z)]\}$ is schlicht and converges, and since $C_{k,n}$ converges to C, we may apply the variation of Carathéodory's theorem mentioned in §5 and find that $f(z)$ is regular at every point of C_k and hence C_k cannot have any ends at $z=\infty$. So the paths of reality of $f(z)$ are the real axis and the star-shaped curves C_k, symmetric about the real axis, intersecting the real axis in $z=\alpha_k$ and $z=\beta_k$ with C_k having no points at $z=\infty$. Consider the domain D bounded by the upper halves of C_j and C_{j+1} and the segments (α_j, α_{j+1}) and (β_j, β_{j+1}). In this domain $\Im(w)$ has the constant sign $(-1)^j$ and hence in $D[w]\neq(-1)^{j+1}i$. The real part of w varies monotonically from $(-1)^{j+1}\infty$ to $(-1)^j\infty$ as z traverses the boundary of D in the positive direction starting from $z=\gamma_{j+1}$. Applying the theorem of Darboux we have: D is mapped schlichtly by $w=f(z)$ onto the half-plane $(-1)^j\Im(w)>0$. Therefore, if $\delta=0$, the surface of the inverse of $w=f(z)$ is a surface of the first class.

Case II: $\delta<0$. We may construct a sequence of rational functions $\{R_n(z)\}$ which converges uniformly to $f(z)$ in any closed bounded region by setting $R_n(z)=z(1+\delta z/\lambda_n)^{\lambda_n}\prod_{k=1}^{n}(1-z/c_k)/(1-z/\gamma_k)$ where $\{\lambda_n\}$ is a sequence of positive integers increasing to infinity and chosen so that $|\lambda_n/\delta|>c_n$. Then $R_n'(z)=(1+\delta z/\lambda_n)^{\lambda_n-1}M_n(z)/N_n(z)$ where $N_n(z)=\prod_{k=1}^{n}(1-z/\gamma_k)^2$ and $M_n(z)$ is a polynomial of degree $2n+1$. By the same argument used in the preceding paragraph we find that $R_n'(z)$ has at least $2n$ zeros at points $\alpha_{k,n}, \beta_{k,n}, k=1, 2, 3, \cdots, n$, with $\beta_{n,n}<\gamma_n<\cdots<\beta_{k,n}<\gamma_k<\cdots<\beta_{1,n}<\gamma_1<0<\alpha_{1,n}<c_1<\alpha_{2,n}<\cdots<\alpha_{n,n}<c_n$. Since $R_n(z)$ is zero when $z=-\lambda_n/\delta$, there is at least one zero of $R_n'(z)$ at a point $z=\epsilon_n$ where $c_n<\epsilon_n<-\lambda_n/\delta$. Since the degree of $M_n(z)$ is $2n+1$ we can write

$$R_n'(z)=(1+\delta z/\lambda_n)^{\lambda_n-1}(1-z/\epsilon_n)\prod_{k=1}^{n}(1-z/\alpha_{k,n})(1-z/\beta_{k,n})/(1-z/\gamma_n)^2.$$

Concerning the paths of reality for $R_n(z)$ we have:

(1) The real axis is a path of reality.

(2) The only points at which two paths of reality can intersect are $z=\alpha_{j,n}$, $z=\beta_{j,n}$ $(j=1, 2, \cdots, n)$, $z=\epsilon_n$, $z=-\lambda_n/\delta$, $z=\infty$.

(3) There is one path of reality intersecting the real axis at each of $z=\alpha_{j,n}$, $z=\beta_{j,n}$, $z=\epsilon_n$ since each of these points is a branch point of order one.

(4) There are λ_n-1 paths of reality intersecting the real axis at $z=-\lambda_n/\delta$.

(5) There are λ_n paths of reality besides the real axis through $z=\infty$.

(6) As in case 1, no two $\beta_{j,n}$ can lie on the same path of reality.

(7) No path of reality can pass through any $\beta_{j,n}$, and $z=\epsilon_n$ for this would result in the intersection of two paths of reality at some point not included in (2).

(8) $\alpha_{n,n}$ and ϵ_n cannot lie on the same path of reality since the real part of $R_n(z)$ is a monotone function as z traces a path from ϵ_n to $\alpha_{n,n}$ along the real axis and from $\alpha_{n,n}$ to ϵ_n along a curve in the upper half-plane.

(9) No $\alpha_{j,n}$, $j\neq n$, can lie on the same path of reality as ϵ_n for the same reason as in (7).

(10) No $\beta_{j,n}$ or $\alpha_{j,n}$ can lie on the same path of reality as $-\lambda_n/\delta$ for the same reason.

(11) As in (8) no path of reality can pass through ϵ_n and $-\lambda_n/\delta$.

(12) No path of reality through any $\alpha_{j,n}$ or $\beta_{j,n}$ can pass through

$z=\infty$, for this would require more than λ_n+1 paths of reality through $z=\infty$ or would contradict one of the statements (1)–(11).

Therefore the paths of reality must consist of n curves $C_{k,n}$, $k=1, 2, \cdots, n$, and λ_n curves $D_{k,n}$, $k=1, 2, \cdots, \lambda_n$. The curves $C_{k,n}$ are simple closed curves, nonintersecting and symmetric about the real axis with $C_{j,n}$ intersecting the real axis in $z=\alpha_{j,n}$, $\beta_{j,n}$. The curves $D_{k,n}$ are simple, nonintersecting, and symmetric about the real axis with $D_{1,n}$ passing through $z=\epsilon_n$ and $z=\infty$, and $D_{k,n}$, $k=2, 3, \cdots, \lambda_n$, passing through $z=-\lambda_n/\delta$ and $z=\infty$.

As $n\to\infty$ the point $z=\epsilon_n$ and the curves $D_{k,n}$ disappear from consideration, for if $D_{k,n}$ had points interior to $|z|<R<\infty$ for infinitely many n, there would be points of $C_{j,n}$ interior to $|z|<R$ for all $j\leqq n$. None of these intersect and in the limit there would be some point $z=z_0$, $|z_0|<R$, $|z_0-\gamma_k|\geqq\rho>0$ for all k, such that in the neighborhood of z_0 there would be segments of infinitely many nonintersecting paths of reality of $f(z)$, which is impossible since $f(z)$ is meromorphic in $|z|<\infty$ and hence holomorphic at $z=z_0$. So the paths of reality of $f(z)$ in case II are the same as in case I, and by the same argument we conclude that if $\delta<0$, the surface of the inverse of $w=f(z)$ is a surface of the first class, which completes the proof of Theorem III.

9. **Partial converse of Theorem II.** By obvious extensions of the methods used in paragraph 8 we can prove

THEOREM IV. *Let $w=f(z)$ be meromorphic in $0<|z|<\infty$ with $f(1)=0$ and $f'(1)>0$. If $w=f(z)$ has the form*

$$f(z) = K(z-1)\exp(\sigma_1 z+\sigma_2/z)\prod_{k=1}^{\infty}(1-z/c_k)(1-c_{-k}/z)/(1-z/\gamma_k)(1-\gamma_{-k}/z)$$

where K is real, σ_1, σ_2 are real and nonpositive, and

$$-\infty<\gamma_{k+1}<\gamma_k<\gamma_{-k}<\gamma_{-k-1}<0<c_{-k-1}<c_{-k}<1<c_k<c_{k+1} \quad (k=1,2,3,\cdots),$$

$$\sum_{k=1}^{\infty}1/c_k<\infty, \qquad \sum_{k=1}^{\infty}c_{-k}<\infty,$$

$$\sum_{k=1}^{\infty}1/|\gamma_k|<\infty, \qquad \sum_{k=1}^{\infty}|\gamma_{-k}|<\infty,$$

then the Riemann surface of the inverse of $w=f(z)$ is a surface of the class described in §3.

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