

ON APPROXIMATION TO ANALYTIC FUNCTIONS BY RATIONAL FUNCTIONS

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Let $f(z)$ be analytic in the interior of a rectifiable Jordan curve C and continuous in the corresponding closed region \bar{C} . The relation between continuity properties of $f(z)$ on C and degree of approximation to $f(z)$ by polynomials $\pi_n(z)$ in z of respective degrees n , $n = 1, 2, \dots$, has been extensively studied. In the present paper we study the relation between continuity properties of $f(z)$ on C and degree of approximation to $f(z)$ on \bar{C} by rational functions $r_n(z)$ in z of respective degrees n whose poles lie exterior to C . We thus deal with what is frequently referred to in the theory as Problem α .

1. Definitions. By a rational function in z of degree n we shall mean a function $r_n(z)$ of the form $r_n(z) = (\sum_{j=0}^n a_j z^j) / (\sum_{j=0}^n b_j z^j)$, where the denominator is not identically zero. None of the a_j is assumed to be different from zero. The degree of a rational function will always be indicated by its subscript.

The continuity properties of a function $f(z)$ to which we approximate will be described in terms of the modulus of continuity of $f(z)$ or one of its derivatives. Let E be a closed bounded set in the z -plane which consists of the mutually disjoint Jordan arcs, Jordan curves, and closed Jordan regions E_j , $j = 1, \dots, \mu$. Let $f(z)$ be a real or complex function that is continuous on each E_j . The modulus of continuity $\omega_j(\delta)$ of $f(z)$ on E_j shall be defined as the maximum of $|f(z_1) - f(z_2)|$ for all z_1 and z_2 in E_j such that $|z_1 - z_2| \leq \delta$. The modulus of continuity $\omega(\delta)$ of $f(z)$ on E shall be defined as the maximum of $\omega_j(\delta)$, $1 \leq j \leq \mu$. For properties of the modulus of continuity, the reader is referred to [1, §2].

Let E with boundary C be a closed limited point set in the z -plane whose complement K is connected and regular in the sense that K possesses a Green's function $G(x, y)$, $z = x + iy$, with pole at infinity. We shall denote by C_R the locus $G(x, y) = \log R > 0$ which lies in K .

If C is any set of closed curves, we denote the sum of their closed interiors by \bar{C} .

The letter k will be reserved for *positive integers* and *zero*. The letters L and M with or without subscripts denote positive constants which may vary from one theorem and its proof to another and may

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depend on the point sets involved, but which are independent of w, z, δ, m , and n . For any function $f(z)$, $f^{(0)} \equiv f(z)$.

2. Degree of approximation. In this section from given continuity properties of a function we shall derive the existence and degree of convergence of a sequence of approximating rational functions having specified poles.

When we refer to a function $f(z)$ as being analytic in several mutually exterior regions, we do not imply that there is any relationship between the values of $f(z)$ for z in different regions. Here and throughout we write "there exist polynomials (rational functions)" rather than "for each $n, n=1, 2, \dots$, there exists a polynomial (a rational function) of degree n ."

THEOREM 2.1. *Let C consist of a finite number of mutually exterior analytic Jordan curves. Let $f(z)$ be analytic in the interior of C and continuous on \bar{C} . Suppose $f^{(k)}(z)$ exists and is continuous on C with modulus of continuity $\omega(\delta)$, $\omega(\delta) \neq 0$. Let points α_{nj} , $j=1, \dots, n$; $n=1, 2, \dots$, be given on or exterior to C_A , $A>1$. Then there exist rational functions $r_n(z)$,*

$$(2.1) \quad r_n(z) = \frac{a_{n0}z^n + a_{n1}z^{n-1} + \dots + a_{nn}}{(z - \alpha_{n1})(z - \alpha_{n2}) \dots (z - \alpha_{nn})},$$

such that

$$|f(z) - r_n(z)| \leq M\omega(1/n)/n^h, \quad z \text{ on } \bar{C}.$$

The α_{nj} need not all be distinct or finite. We make the convention that if an α_{nj} is infinite, then the factor $(z - \alpha_{nj})$ is to be replaced by unity. It will appear from the proof of Theorem 2.1 that the approximating rational functions $r_n(z)$ can be chosen so that $\sum_{i=0}^n a_{ni}\alpha_{nj}^{n-i} \neq 0$, $j=1, \dots, n$. It is sufficient to assume the existence of $f^{(k)}(z)$ on the boundary merely in the one-dimensional sense.

Theorem 2.1 for the case that C is a single analytic Jordan curve and $f^{(k)}(z)$ satisfies a Lipschitz condition has been stated by Sewell [2, Theorem 4.3]; his proof, however, contains an error. It is impossible to choose the quantities involved so that inequality (4.13) in Sewell's paper is satisfied.

We proceed to prove Theorem 2.1. It is known [1, Theorem 3.3] that there exist polynomials $\pi_m(z)$ in z such that

$$(2.2) \quad |f(z) - \pi_m(z)| \leq M_1\omega(1/m)/m^h, \quad z \text{ in } \bar{C}.$$

Let β_{nj} , $j=1, \dots, n+1$, be $n+1$ points on C such that the maximum of the absolute value of

$$V_n(z_1, z_2, \dots, z_{n+1})$$

$$= \left[\prod_{i,j=1, i < j}^{n+1} (z_i - z_j) \right] / \left[\prod_{i,j=1}^{i=n+1, j=n} (z_i - \alpha_{nj}) \right], \quad n \geq 2,$$

for the $z_j, j=1, \dots, n+1$, on C is taken on for $z_j = \beta_{nj}$. If any α_{nj} is at infinity, the $n+1$ factors $(z_i - \alpha_{nj})$ are to be replaced by unity. The β_{nj} are the Fekete-Shen points.

Let z lie on C ; for t exterior to C it is easily verified that

$$(2.3) \quad r_n(t, z) = \sum_{j=1}^{n+1} \frac{1}{t - \beta_{nj}} \frac{V_n(\beta_{n1}, \dots, \beta_{n,j-1}, z, \beta_{n,j+1}, \dots, \beta_{n,n+1})}{V_n(\beta_{n1}, \dots, \beta_{n,n+1})}$$

is a function of form (2.1) which interpolates to $1/(t-z)$ in the points β_{nj} . Choose $R, 1 < R < A$. It is a consequence of (2.3) and the choice of the β_{nj} that $|r_n(t, z)| \leq (n+1)/d$ for z on C and t on C_R , d denoting the minimum distance from C to C_R . Furthermore [3, p. 185]

$$\frac{1}{t-z} - r_n(t, z) = \frac{1}{t-z} \frac{\omega_n(z)}{\omega_n(t)},$$

where $\omega_n(z) = (z - \beta_{n1}) \dots (z - \beta_{n,n+1}) / (z - \alpha_{n1}) \dots (z - \alpha_{nn})$. Hence

$$\left| \frac{1}{t-z} \frac{\omega_n(z)}{\omega_n(t)} \right| \leq \frac{n+2}{d}, \quad z \text{ on } C, t \text{ on } C_R.$$

Let $G(z, \alpha_{nj})$ denote Green's function for the exterior of C_R with pole at $z = \alpha_{nj}$. Then $w = \phi(z, \alpha_{nj}) = \exp [G(z, \alpha_{nj}) + iH(z, \alpha_{nj})]$, where H is conjugate to G in the exterior of C_R , maps the exterior of C_R onto $|w| > 1$ so that the point $z = \alpha_{nj}$ goes into the point at infinity in the w -plane. On C_R , $|\phi(z, \alpha_{nj})| = 1$. Hence

$$(2.4) \quad \left| \frac{\omega_n(z)}{\omega_n(t)} \frac{\prod_{j=1}^n \phi(t, \alpha_{nj})}{t-z} \right| \leq \frac{n+2}{d}, \quad z \text{ on } C, t \text{ on } C_R.$$

The functions $[\prod_{j=1}^n \phi(t, \alpha_{nj})] / [(t-z)\omega_n(t)]$, considered as functions of t , for t exterior to C_R and z on C , are analytic and their moduli are single-valued. Thus (2.4) is valid for z on C and t exterior to C_R .

There exist constants L_j such that $|\phi(t, \alpha_{nj})| \geq L_j > 1$ for t on C_A . Set $L = \min_{1 \leq j \leq n} L_j$. Then

$$(2.5) \quad \left| \frac{\omega_n(z)}{\omega_n(t)} \frac{1}{t-z} \right| \leq \frac{n+2}{dL^n}, \quad z \text{ on } C, t \text{ on } C_A.$$

If $R_{n,m}(z)$ is the rational function of form (2.1) which interpolates to $\pi_m(z)$ at the points β_{nj} , $j=1, \dots, n+1$, we have [3, p. 186]

$$(2.6) \quad \pi_m(z) - R_{n,m}(z) = \frac{1}{2\pi i} \int_{C_A} \frac{\omega_n(z)}{\omega_n(t)} \frac{\pi_m(t)}{t-z} dt, \quad z \text{ on } C.$$

From (2.2) it follows that the $|\pi_m(z)|$ are uniformly bounded for z on C and hence [3, p. 77] for z on C_A we have $|\pi_m(z)| \leq M_2 A^m$. Thus from (2.5), (2.6) we obtain

$$|\pi_m(z) - R_{n,m}(z)| \leq M_3 n A^m / L^n, \quad z \text{ on } \bar{C}.$$

Let q be an integer such that $A/L^q = r$, $r < 1$. Then for $j=0, 1, \dots, q-1$; $m=1, 2, \dots$, it follows that

$$|\pi_m(z) - R_{qm+j,m}(z)| \leq M_4 m r^m, \quad z \text{ on } C.$$

We have $r^m \leq M_5 / m^{k+2}$ and [1, Theorem 3.1] $1/m^{k+2} \leq M_6 \omega(1/m)/m^{k+1}$. Hence with the aid of (2.2) we obtain

$$|f(z) - R_{qm+j,m}(z)| \leq M_7 \omega(1/m)/m^k, \quad z \text{ on } \bar{C}.$$

Since [1, Theorem 3.1] there exists a constant M_8 such that $\omega(1/m)/m^k < M_8 \omega(1/(qm+j))/m^k$, the theorem follows if we define $r_n(z) = 1/(z-\alpha_{n1}) \cdots (z-\alpha_{nn})$ for $n=1, 2, \dots, q-1$ and $r_{qm+j}(z) = R_{qm+j,m}(z)$ for $j=0, 1, \dots, q-1$; $m=1, 2, \dots$.

In the case that C is a single analytic Jordan curve, the above proof can be somewhat simplified by using a known lemma [3, p. 250].

3. Degree of approximation; inverse problem. We turn now to the converse question: given a sequence of rational functions which converge to a function in a certain manner, what can be said about the continuity properties of the function? We establish our results for a single closed Jordan region, but they clearly hold for a finite number of such regions which are mutually exterior.

DEFINITION 3.1. A curve C is said to be of Type D if (i) C is a rectifiable Jordan curve such that for every arc of C the ratio of the length of that arc to the length of its corresponding chord is less than a constant A_0 ; and (ii) there exists a number $\delta_0 > 0$ such that through any point P of C there is a circle of radius δ_0 whose closed interior lies in \bar{C} .

To establish our results, we approximate by polynomials the rational functions which are assumed to converge to $f(z)$ in a certain manner; a known theorem on polynomial approximation then enables us to draw the desired conclusions.

We henceforth assume $\Omega(x)$ to be a real, non-negative function which is nonincreasing for x sufficiently large and which is such that

$\int^\infty [\Omega(x)/x] dx$ exists.

The following known result on polynomial approximation [1, Theorem 5.3] will be utilized:

THEOREM 3.1. *Let $f(z)$ be defined in \bar{C} , where C is a curve of Type D; let polynomials $\pi_n(z)$ in z exist such that*

$$(3.1) \quad |f(z) - \pi_n(z)| \leq \Omega(n)/n^h, \quad z \text{ in } \bar{C}.$$

Then $f(z)$ is analytic in the interior of C and continuous on \bar{C} ; $f^{(k)}(z)$ exists and is continuous on \bar{C} with modulus of continuity $\omega(\delta)$ which satisfies the condition

$$(3.2) \quad \omega(\delta) \leq L \left[\delta \int_a^{a/\delta} \Omega(x) dx + \int_{1/\delta}^\infty \frac{\Omega(x)}{x} dx \right], \quad 0 < \delta \leq 1/a,$$

where $a > 1$ is a constant independent of δ .

For the case $k=0$, Theorem 3.1 is formulated somewhat differently in [1], but the method of proof employed there can be used to establish the theorem as it is stated above. Inequalities (5.4) and (5.6) in [1, Theorem 5.3] hold as before. The restriction that $q_n(\xi) = v(\xi)$ of [1, Theorem 5.3] is not necessary for the case $k=0$. Under the hypothesis of the present Theorem 3.1, an inequality corresponding to (5.7) of [1, Theorem 5.3], where $p_n(z)$ is replaced by $\pi_n(z)$, is immediately established, and the theorem follows as before.

It is a consequence of the proof of Theorem 3.1 given in [1] that the only restriction on the constant a is that $\Omega(x)$ be nonincreasing for $x > a$. The method used to prove Theorem 3.1 is a modification of one employed by de la Vallée Poussin to establish results on trigonometric approximation.

We prove now

THEOREM 3.2. *Let $f(z)$ be defined in \bar{C} , where C is a curve of Type D; let rational functions $r_n(z)$ all of whose poles lie on or exterior to C_A , $A > 1$, exist such that*

$$(3.3) \quad |f(z) - r_n(z)| \leq \Omega(n)/n^h, \quad z \text{ in } \bar{C}.$$

Then the conclusion of Theorem 3.1 is valid.

Theorem 3.2 has been established by Sewell [2, Theorem 4.2] for the special case that C is an analytic Jordan curve and $\Omega(n) = Ln^{-\alpha}$, $0 < \alpha \leq 1$.

Choose R_1 and R_2 such that $1 < R_1 < R_2 < A$. Let $P_{m,n}(z)$ be a polynomial in z of degree m which interpolates to $r_n(z)$ in $m+1$ equally-

distributed points z_j on C_{R_1} . (The " $m+1$ equally-distributed points on C_{R_1} " are the images of the $(m+1)$ st roots of unity under the conformal map of $|w| > 1$ onto the exterior of C_{R_1} by the function $w = \phi(z)/R_1$.) The Lagrange-Hermite interpolation formula yields

$$r_n(z) - P_{m,n}(z) = \frac{1}{2\pi i} \int_{C_{R_2}} \frac{\omega_m(z)r_n(t)}{\omega_m(t)(t-z)} dt, \quad z \text{ on } C_{R_1},$$

where $\omega_m(z) = (z-z_1) \cdots (z-z_{m+1})$. For z on C_{R_1} and t on $C_{R_2} = (C_{R_1})_{R_2/R_1}$ we have [4, Theorem 4.7] that $|\omega_m(z)/\omega_m(t)| \leq M(R_1/R_2)^m$. It is a consequence of the uniform boundedness of $|r_n(z)|$ and a known result [3, p. 250] that

$$|r_n(z)| \leq M_1[(AR_2 - 1)/(A - R_2)]^n$$

for z on C_{R_1} . Hence

$$(3.4) \quad |r_n(z) - P_{m,n}(z)| \leq M_2 \left(\frac{R_1}{R_2} \right)^m \left(\frac{AR_2 - 1}{A - R_2} \right)^n, \quad z \text{ on } \bar{C}.$$

Let q be an integer such that $[(AR_2 - 1)/(A - R_2)](R_1/R_2)^q = r < 1$. It then follows from (3.3) and (3.4) that for $j=0, 1, \dots, q-1$; $n=1, 2, \dots$,

$$|f(z) - P_{qn+j,n}(z)| \leq \Omega(n)/n^k + M_2 r^n, \quad z \text{ on } \bar{C}.$$

Since $r^n < M_2/n^{k+2}$ and $\Omega(n) \leq \Omega((qn+j)/2q)$ for n sufficiently large, we have

$$|f(z) - P_{qn+j,n}(z)| \leq M_4 [\Omega((qn+j)/2q) + 1/(qn+j)^2] / (qn+j)^k, \quad z \text{ on } \bar{C}.$$

If we define the polynomials $\pi_n(z)$ of Theorem 3.1 as $\pi_n(z) \equiv 0$ for $n=1, \dots, q-1$; $\pi_{qh+j}(z) \equiv P_{qh+j,h}(z)$ for $j=0, 1, \dots, q-1$; $h=1, 2, \dots$, we obtain (3.1) where the $\Omega(x)$ of (3.1) is in terms of the present $\Omega(x)$, $M_5[\Omega(x/2q) + x^{-2}]$. Hence it follows from Theorem 3.1 that $f(z)$ is analytic interior to C , continuous on \bar{C} , and $f^{(k)}(z)$ exists and is continuous on \bar{C} with modulus of continuity $\omega(\delta)$ which satisfies the condition

$$\omega(\delta) \leq L \left\{ \delta \int_a^{a/\delta} [\Omega(x/2q) + x^{-2}] dx + \int_{1/\delta}^\infty [\Omega(x/2q)/x + x^{-2}] dx \right\}$$

for $0 < \delta \leq 1/a$, where a is such that $a > 2q$ and $\Omega(x)$ is increasing for $x > a/2q$. On suitable change of variable and simplification we obtain an inequality of form (3.2).

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A CONVEX METRIC WITH UNIQUE SEGMENTS

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1. **Introduction.** If $D(x, y)$ is a convex metric for a continuous curve M (compact, locally connected, metric continuum), then for each pair of points p, q of M there is an arc pq in M from p to q such that pq is isometric to a straight line interval. We shall call it a segment.

Each continuous curve has a convex metric $[1; 3; 5; 6; 2]$. However, if $D(x, y)$ is a convex metric for M , there may be two segments from p to q . If $M = S_2$ is the surface of a sphere, $D(x, y)$ is the size of the central angle subtended by x and y , and p, q are diametrically opposite points, then there are many segments from p to q . In fact, we show in §4 that if $D(x, y)$ is any convex metric whatsoever for S_2 , each point of S_2 belongs to a pair of points which are not joined by a unique segment.

There is a dense subset W of S_2 such that no two points of W are diametrically opposite. If $D(x, y)$ is the previously mentioned convex metric for S_2 , then each pair of points of W are joined by one and only one segment. We shall show that for any continuous curve there is such a dense subset and such a convex metric.

THEOREM. *Each continuous curve M has a dense subset W and a convex metric $D(x, y)$ such that each pair of points of W belongs to a unique segment.*

We shall prove this result in much the same manner that it was shown that any continuous curve can be convexified. The metric is

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