KERNEL FUNCTIONS AND CLASS L^2

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1. **Introduction.** Let $f(z)$ be a measurable function of the real variables x and y in the plane of the complex $z(=x+iy)$ variable. Denote by B a finite domain bounded by a finite number of analytic Jordan curves, and by $L^2(B)$ the Hilbert space of functions $f(z)$ of summable square on B, i.e.,

$$\|f\|_B^2 = \iint_B dxdy\,|f(z)|^2.$$

The subspace of functions of $L^2(B)$ which are analytic on B is designated by $\mathcal{L}^2(B)$. Associated with B is the classical reproducing kernel $K(z, \zeta^*)$ [1], and it is well known that if $f_1(z)$ belongs to $\mathcal{L}^2(B)$, then

$$f_1(z) = \iint_B d\xi d\eta K(z, \zeta^*) f_1(\zeta) \qquad (\zeta = \xi + i\eta). \tag{1.1}$$

(* denotes the complex conjugate.) Also there is a complete orthonormal set of functions on B which span $\mathcal{L}^2(B)$ and for which there exists a Riesz-Fischer theory with convergence in L^2 norm. It follows immediately (Walsh [3, pp. 149–151]; Bergman [1, pp. 5–10]) that (1.1) may be applied to functions $f(z)$ in $L^2(B)$, and the result is the prcjection $f_1(z)$ of $f(z)$ on $\mathcal{L}^2(B)$. That is,

$$f_1(z) = \iint_B d\xi d\eta f(\zeta) K(z, \zeta^*) \qquad (\text{for } z \text{ in } B).$$

Recently, Bergman and Schiffer [2] have introduced to the theory of kernel functions the new kernel

$$L(z, \zeta) = -\frac{2}{\pi}\frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta}.$$

$g(z, \zeta)$ is the Green's function for B, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

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It is shown that

$$L(z, \zeta) = \frac{1}{\pi(z - \zeta)^2} - l(z, \zeta),$$

where $l(z, \zeta)$ is a single-valued analytic function in both variables on the closure of B. In addition, if $f_1(z)$ is in $\mathcal{L}^2(B)$ and if

$$P. \iint_B d\xi d\eta L(z, \zeta)[f_1(\zeta)]^* = \lim_{\epsilon \to 0} \iint_{B: |z-\zeta|>\epsilon} d\xi d\eta L(z, \zeta)[f_1(\zeta)]^*,$$

then Bergman and Schiffer have proved that

$$(1.2) \qquad P. \iint_B d\xi d\eta L(z, \zeta)[f_1(\zeta)]^* = 0.$$

Next let $f_2(z)$ be determined by the relation

$$(1.3) \qquad f(z) = f_1(z) + f_2(z).$$

Then $f_2(z)$ belongs to the subspace which is the orthogonal complement to $\mathcal{L}^2(B)$. Indeed,

$$\iint_B d\xi d\eta f_1(\zeta)[f_2(\zeta)]^* = 0.$$

It is the purpose of this paper to study the relationship of the L-kernel to $L^2(B)$. Just as the K-kernel determines the projection $f_1(z)$ of $f(z)$ on $\mathcal{L}^2(B)$, so the L-kernel determines the projection $f_2(z)$ of $f(z)$ on the orthogonal complement to $\mathcal{L}^2(B)$. In fact, if we define $f(z) = 0$ exterior to B and take

$$L(f) = \underset{\epsilon \to 0}{\text{l.i.m.}} \iint_{B: |\zeta - z|>\epsilon} d\xi d\eta L(z, \zeta)[f(\zeta)]^*,$$

then $L[L(f)] = f_2(z)$ almost everywhere in B. In addition, it is found that the subspace of functions which satisfy a Lipschitz condition of order α, $0 < \alpha < 1$, on B is invariant under the L-transformation.

2. **A theorem of Beurling.** In private conversation with this author, Beurling has communicated the following result.

THEOREM. *If* (a) $f(z) \in L^2$ *(over the complex plane),*

$$\text{(b)} \qquad g_\epsilon(z) = \frac{1}{\pi} \iint_{|\zeta - z|>\epsilon} d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2},$$

then there exists a function $g(z)$ *in* L^2 *such that*

$$\text{(a)} \qquad g(z) = T(f) = \underset{\epsilon \to 0}{\text{l.i.m.}} \frac{1}{\pi} \iint_{|\zeta - z| > \epsilon} d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2},$$

$$\text{(b)} \qquad \{T[T(f)]^*\}^* = T[T(f^*)]^* = f(z) \text{ a.e.},$$

$$\text{(c)} \qquad \iint dxdy \, |f(z)|^2 = \iint dxdy \, |g(z)|^2.$$

Unless otherwise indicated, the range of integration here and in what follows will be the entire complex plane. To prove the theorem, one first shows the existence of the Fourier-Plancherel transform of $g(z)$. Indeed, the two-dimensional transform of $g(z)$ is $(-z^2/2\pi|z|^2)$ $F(x,y)$ where $(1/2\pi)F(x,y)$ is the transform of $f(\zeta)$. Application of this to the inversion formulae in the statement of the theorem yields the proof.

3. $L(z,\zeta)$ **and the class** $L^2(B)$. Let $f(z)$ be any function of $L^2(B)$. We define $f(z)=0$ exterior to B. It is evident from the Schwarz inequality that $\iint_B d\xi d\eta l(z, \zeta)[f(\zeta)]^*$ converges absolutely. From this and Beurling's theorem, one now finds that a function $g_2(z)$ of summable square on B exists such that

$$(3.1) \qquad \lim_{\epsilon \to 0} \left\| g_2(z) - \frac{1}{\pi} \iint_{B:|\zeta - z| > \epsilon} d\xi d\eta L(z, \zeta) [f(\zeta)]^* \right\|_B = 0.$$

Now $f_2(z)$, as defined by (1.3), is orthogonal to each member of $\mathcal{L}^2(B)$. Therefore we may conclude that

$$(3.2) \qquad \iint_B d\xi d\eta l(z, \zeta)[f_2(\zeta)]^* = 0$$

for each z in B. It now follows from (1.2), (3.1) and (3.2) that

$$(3.3) \qquad \lim_{\epsilon \to 0} \left\| g_2(z) - \frac{1}{\pi} \iint_{B:|\zeta - z| > \epsilon} d\xi d\eta \frac{[f_2(\zeta)]^*}{(\zeta - z)^2} \right\|_B = 0.$$

Accordingly, it is consistent to define $g_2(z)$ in the entire complex plane by means of (3.3), for it reduces to (3.1) when z is in B. When z is exterior to B, $g_2(z)$ vanishes since the kernel becomes an analytic function of ζ on B. Next we prove a

LEMMA. *If z is in B, then*

$$\iint_B d\xi d\eta K(z, \zeta^*) g_2(\zeta) = 0.$$

For each positive ϵ, $(w = u + iv)$,

$$\iint_{B:|\zeta-w|>\epsilon} dudv \frac{[f_2(w)]^*}{(\zeta - w)^2}$$

is of summable square on B, as is $K(z, \zeta^*)$ with respect to ζ for each fixed z in B. We replace $g_2(z)$ in the lemma by its value (3.3). Then it follows from a well known theorem for sequences of functions convergent in the mean that

$$\iint_B d\xi d\eta K(z, \zeta^*) \frac{1}{\pi} \underset{\epsilon\to 0}{\text{l.i.m.}} \iint_{B:|\zeta-w|>\epsilon} dudv \frac{[f_2(w)]^*}{(\zeta - w)^2}$$
$$= \lim_{\epsilon\to 0} \iint_B d\xi d\eta K(z, \zeta^*) \frac{1}{\pi} \iint_{B:|\zeta-w|>\epsilon} dudv \frac{[f_2(w)]^*}{(\zeta - w)^2}.$$

The integrals on the right are absolutely convergent for each z in B, so we may interchange the order of integration. This produces

$$\lim_{\epsilon\to 0} \iint_B dudv [f_2(w)]^* \frac{1}{\pi} \iint_{B:|w-\zeta|>\epsilon} d\xi d\eta \frac{K(z, \zeta^*)}{(w - \zeta)^2}$$
$$= \iint dudv [f_2(w)]^* \frac{1}{\pi} \underset{\epsilon\to 0}{\text{l.i.m.}} \iint_{B:|w-\zeta|>\epsilon} d\xi d\eta \frac{K(z, \zeta^*)}{(w - \zeta)^2}. \tag{3.4}$$

It is a result of Bergman and Schiffer [2, p. 216] that

$$l(z, w) = \frac{1}{\pi} P. \iint_B d\xi d\eta [K(\zeta, w^*)]^* \frac{1}{(\zeta - z)^2},$$

and the existence of this principal value implies that it is the same as the result obtained by taking the l.i.m. Inasmuch as $[K(\zeta, w^*)]^* = K(w, \zeta^*)$, it now follows that the right member of (3.4) is

$$\iint_B dudv [f_2(w)]^* l(z, w).$$

Since this is zero, the lemma is proved.

We may now state

THEOREM 3.1. *If $f(z) \in L^2(B)$, then*

$$L[L(f)] = f_2(z) \quad a.e. \quad in\ B.$$

If we define $f_2(z) = 0$ for z exterior to B, then we may write $g_2(z) = T(f_2^*)$. It follows from Beurling's theorem that $T(g_2^*) = f_2(z)$. Since $g_2(z) = 0$ for z exterior to B, we have

$$T(g_2^*) = \underset{\epsilon\to 0}{\text{l.i.m.}} \frac{1}{\pi} \iint_{B:|z-\zeta|>\epsilon} d\xi d\eta \frac{[g_2(\zeta)]^*}{(z - \zeta)^2}.$$

But the lemma implies that $g_2(z)$ belongs to the orthogonal complement to the space $\mathfrak{L}^2(B)$. Therefore

$$L(g_2) = -\iint_B d\xi d\eta l(z, \zeta)[g_2(\zeta)]^* + \underset{\epsilon\to 0}{\text{l.i.m.}} \frac{1}{\pi}\iint_{B:|z-\zeta|>\epsilon} d\xi d\eta \frac{[g_2(\zeta)]^*}{(z-\zeta)^2}$$

$$= T(g_2^*) = f_2(z).$$

Since $L(g_2) = L[L(f)]$, the theorem follows.

COROLLARY. $\|f_2\|_B = \|g_2\|_B$.

COROLLARY. *If* $f(z) \in L^2(B)$, *then* $L[L(f)] = 0$ *a.e. in B if and only if* $f(z) \in \mathfrak{L}^2(B)$.

4. Invariance of the class of functions Lip α on B. We first consider the class of functions which satisfy a Lipschitz condition uniformly over the complex plane.

THEOREM 4.1. *If* (a) $\iint dxdy|f(z)|^2 < \infty$,
(b) $|f(z_1) - f(z_2)| \leq A|z_1 - z_2|^\alpha$ *when* $|z_1 - z_2| < \delta$,
(c) $0 < \alpha < 1$,
then $g(z)(=T(f))$ *satisfies a Lipschitz condition of order* α.

Under the conditions stated, the principal value integral

$$\frac{1}{\pi} P. \iint d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2}$$

exists. Inasmuch as

$$\frac{1}{\pi} P. \iint_{|\zeta - z|<\delta} d\xi d\eta \frac{1}{(\zeta - z)^2} = 0,$$

it follows that

$$\frac{1}{\pi} P. \iint d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2} = \frac{1}{\pi}\iint_{|\zeta - z|>\delta} d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2} + \frac{1}{\pi}\iint_{|\zeta - z|<\delta} d\xi d\eta \frac{f(\zeta) - f(z)}{(\zeta - z)^2}.$$

We note that the last integral on the right is absolutely convergent because of the Lipschitz condition on $f(z)$. The first integral on the right also converges, and thus the principal value integral exists.

It is also evident that the conditions of the theorem imply that $f(z)$ is continuous and bounded in the z-plane, tending uniformly to zero as $|z| \to \infty$.

Now consider any two points z_1 and z_2 such that $|z_1-z_2|<\delta$. A short calculation yields

$$g(z_1) - g(z_2) \tag{4.1}$$
$$= \frac{2(z_1 - z_2)}{\pi} P. \iint d\xi d\eta [f(\zeta) - f(z_1)] \frac{\zeta - (z_1 + z_2)/2}{(\zeta - z_1)^2(\zeta - z_2)^2}.$$

Here $P.$ indicates that the principal value is to be calculated with respect to both singularities of the integrand. We observe that the region C_δ: $|\zeta-z_2|\leqq 2\delta$ contains C_{12}: $|\zeta-z_2|\leqq 2|z_1-z_2|$, and we denote by M and N the regions given respectively by M: $\{|\zeta-z_2|\leqq 2\delta, |\zeta-z_2|>2|z_1-z_2|\}$ and N: $\{|\zeta-z_2|\leqq 2|z_1-z_2|\}$. If ζ belongs to M, then it may be verified from the geometry that

$$\frac{1}{2} \leqq \left|\frac{\zeta - z_2}{\zeta - z_1}\right| \leqq 2, \qquad \frac{1}{2} \leqq \left|\frac{\zeta - (z_1 + z_2)/2}{\zeta - z_2}\right| \leqq 2. \tag{4.2}$$

Designate by I, II, and III respectively those parts of the right side of (4.1) taken over the regions M, N, and $\overline{C}_\delta$: $|\zeta-z_2|>2\delta$. Using the inequalities (4.2) and the Lipschitz condition for $f(z)$, the integral I is in absolute value less than or equal to

$$\frac{2^{4+\alpha}A\,|z_1 - z_2|}{\pi} \iint_M d\xi d\eta\, |\zeta - z_2|^{\alpha-3}$$
$$= \frac{2^{4+2\alpha}A}{1 - \alpha} [\,|z_1 - z_2|^\alpha - \delta^{\alpha-1}|z_1 - z_2|\,]$$
$$= O(|z_1 - z_2|^\alpha).$$

We rewrite the integral II in the form

$$\frac{1}{\pi} P. \iint_N d\xi d\eta \left[\frac{f(\zeta) - f(z_1)}{(\zeta - z_1)^2} - \frac{f(\zeta) - f(z_2)}{(\zeta - z_2)^2} + \frac{f(z_2) - f(z_1)}{(\zeta - z_2)^2}\right].$$

The third of these integrals is zero, and the first two converge absolutely in accordance with the Lipschitz condition for $f(z)$. An estimate for each of the first two integrals is readily obtained. We have

$$\left|\iint_N d\xi d\eta \frac{f(\zeta) - f(z_1)}{(\zeta - z_1)^2}\right| \leqq A \iint_{|\zeta-z_1|\leqq 3|z_1-z_2|} d\xi d\eta\, |\zeta - z_1|^{\alpha-2}$$
$$= \frac{2\pi A}{\alpha} 3^\alpha |z_1 - z_2|^\alpha$$
$$= O(|z_1 - z_2|^\alpha).$$

The estimate is uniform for all z_1 and z_2 for which $|z_1-z_2|<\delta$. The same estimate may be obtained for the second integral.

If ζ is in $\overline{C}_\delta$, the inequalities (4.2) are still valid. Using these, we may estimate the integral III. Indeed, in absolute value, it is less than or equal to

$$\frac{16|z_1-z_2|}{\pi}\iint_{\overline{C}_\delta} d\xi d\eta \frac{|f(\zeta)|}{|\zeta-z_2|^3}$$
$$\leq \frac{16|z_1-z_2|}{\pi}\left[\iint d\xi d\eta |f(\zeta)|^2 \iint_{\overline{C}_\delta} d\xi d\eta \frac{1}{|\zeta-z_2|^6}\right]^{1/2}$$
$$= O(|z_1-z_2|)$$

uniformly for all z_1 and z_2 for which $|z_1-z_2|<\delta$. Thus it follows that (4.1) is uniformly $O(|z_1-z_2|^\alpha)$, and the theorem is proved.

It is important to note that the Lipschitz condition on $f(z)$ need not hold uniformly. For example, if $f(z)$ satisfies a Lipschitz condition of order α at z_0, then $g(z)$ satisfies the same order Lipschitz condition at z_0. With this remark, we have immediately:

THEOREM 4.2. *Let $f(z)$ belong to $L^2(B)$ and satisfy a Lipschitz condition of order α at a point z_0 in B. Then $L(f)$ satisfies the same order Lipschitz condition at z_0. Moreover, if $f(z)$ satisfies a Lipschitz condition of order α uniformly on B, so does $L(f)$.*

It follows from the previous remarks that the part of $L(f)$ which results from the singular kernel satisfies a Lipschitz condition of order α. The part which comes from $l(z, \zeta)$ satisfies a Lipschitz condition of order 1 because of the analyticity of $l(z, \zeta)$.

5. $L(z,\zeta)$ **and the unit circle.** For the unit circle,

$$L(z, \zeta) = \frac{1}{\pi(z-\zeta)^2}, \qquad K(z, \zeta^*) = \frac{1}{\pi(1-\zeta^* z)^2}\cdot$$

While the general results already obtained easily reduce to the special case of the unit circle, the simplicity of $L(z, \zeta)$ leads to special results which may be of some interest.

Let $f(z)$ be a complex-valued function on B: $|z|<1$ such that

$$\iint_{|z|<1} dxdy |f(z)|^2 < \infty.$$

Define $f(z)$ and its two components $f_1(z)$ and $f_2(z)$, as given by (1.3), to be zero for $|z|>1$. We consider the transformation

$$\mathcal{K}(f) = \underset{\epsilon\to 0}{\text{l.i.m.}} \frac{1}{\pi}\iint_{|\zeta-1/z^*|>\epsilon} d\xi d\eta \frac{f(\zeta)}{(1-\zeta^* z)^2}.$$

When $|z|<1$, it follows immediately from the classical kernel theory that $\mathcal{K}(f)=f_1(z)$. On the other hand, if $|z|>1$, we have $\mathcal{K}(f)=\mathcal{K}(f_1)+\mathcal{K}(f_2)$. Application of Green's theorem with respect to the region B_ϵ: $\epsilon<|\zeta-1/z^*|<1$ shows that $\mathcal{K}(f_1)=0$, and from (3.3), we find that $\mathcal{K}(f_2)=1/z^2[g_2(1/z^*)]^*$. Then we have

$$\mathcal{K}(f) = \frac{1}{z^2}\left[g_2\left(\frac{1}{z^*}\right)\right]^*, \qquad \text{(for } |z|>1).$$

To compute $\mathcal{K}^2(f)$, we first observe that $g_2(1/z)=0$ for $|z|<1$, and $f_1(z)=0$ for $|z|>1$. Then

$$\mathcal{K}^2(f) = f_1(z) + \mathcal{K}\left[\frac{1}{z^2}\left[g_2\left(\frac{1}{z^*}\right)\right]^*\right] \qquad \text{a.e.}$$

The last member on the right side is

$$\underset{\epsilon\to 0}{\text{l.i.m.}} \frac{1}{\pi}\iint_{|\zeta-1/z^*|>\epsilon} d\xi d\eta \frac{1/\zeta^2[g_2(1/\zeta^*)]^*}{(1-\zeta^* z)^2}$$

$$= \underset{\epsilon\to 0}{\text{l.i.m.}} \frac{1}{\pi}\iint_{|(w-z)/wz|>\epsilon} dudv \frac{[g_2(w)]^*}{(w-z)^2}$$

since the Jacobian of the transformation $\zeta=1/w^*$ is $1/|w|^4$. It can be shown that this last integral is equal to

$$\underset{\epsilon\to 0}{\text{l.i.m.}} \frac{1}{\pi}\iint_{|w-z|>\epsilon} dudv \frac{[g_2(w)]^*}{(w-z)^2},$$

whence it follows that $\mathcal{K}^2(f)=f_1(z)+[T^*(g_2)]^*=f_1(z)+f_2(z)$ a.e. Then our result is that

$$\mathcal{K}^2(f) = f(z) \qquad \text{a.e.}$$

Bibliography

1. S. Bergman, *The kernel function and conformal mapping*, Mathematical Surveys, No. 5, New York, 1950.
2. S. Bergman and M. Schiffer, *Kernel functions and conformal mapping*, Compositio Math. vol. 8 (1951) pp. 205–249.
3. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, New York, 1950.

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