THE MULTIPLICITY OF A CLASS OF PERFECT SETS

PAUL CIVIN AND H. E. CHRESTENSON

1. **Introduction.** Let P denote the perfect set of measure zero constructed in the following manner: From $[0, 2\pi]=\rho_1^0$ remove an open interval d_1^1, leaving intervals ρ_1^1 and ρ_2^1. At the mth stage of the construction remove d_i^m from ρ_i^{m-1} and call the remaining intervals ρ_{2i-1}^m and ρ_{2i}^m, $i=1, \cdots, 2^{m-1}$. Let

$$\epsilon_m = \sup_{i=1,\cdots,2^{m-1}} \left(\frac{d_i^m}{\rho_i^{m-1}}\right)$$

and

$$\theta_m = \sup_{i=1,\cdots,2^{m-1}} \left(\frac{\rho_{2i}^m}{\rho_{2i-1}^m}, \frac{\rho_{2i-1}^m}{\rho_{2i}^m}\right).$$

N. K. Bari [1] established that if (i) $\epsilon_m=o(1)$ and (ii) $\theta_m=O(1)$, then P is a set of multiplicity for trigonometric series. She further conjectured that the hypothesis (ii) was superfluous. Subsequently S. Verblunsky [3] introduced a lemma upon which he based a proof of the conjecture of Bari. The identical proof was recently repeated in Bari's tract on *The uniqueness problem of the representation of functions by trigonometric series* [2].

Unfortunately, Verblunsky's lemma is not true. We present here a counter example to the lemma and establish a theorem intermediate to the Bari theorem and conjecture.

For the sake of brevity we assume that the notation and construction used by Bari and Verblunsky are known [2, pp. 29–33; 3, pp. 290–294].

2. Verblunsky's lemma states that if $\rho_{2j-1}^k \in R_m'$ and is to the left of d_1^1, and if $\rho_{2j-1}^k < 2\pi(\lambda-1)\lambda^{-m}$, $1<\lambda<2$, then ρ_{2j}^k can be represented as the sum of

(a) a segment $\rho_u^s \in R_m'$, of length $\geqq 2\pi\lambda^{-m}$, plus

(b) a sum of pairs of adjacent ρ_v^t, d_w^t such that

$$\sum (\rho_v^t + d_w^t) < (\lambda - 1)\rho_u^s, \qquad \rho_v^t \in R_m'.$$

Suppose ρ_{2j-1}^k satisfies the hypotheses of the lemma and suppose

(i) $$\rho_{2j}^k \notin R_m',$$

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(ii) $$\rho_{4j}^{k+1} = \rho_{4j-1}^{k+1},$$

(iii) $$\rho_{4j}^{k+1} \in R'_m, \qquad \rho_{4j-1}^{k+1} \in R'_m.$$

It is clear that perfect sets exist for which these conditions are satisfied for arbitrarily large m. These conditions require that d_j^k be located close to the left end of ρ_j^{k-1} and that ρ_{2j}^k be divided symmetrically by an appropriate d_{2j}^{k+1}.

The conclusion of the lemma now is

$$\rho_{2j}^{k} = \rho_{4j}^{k+1} + (\rho_{4j-1}^{k+1} + d_{2j}^{k+1})$$

where

$$\rho_{4j-1}^{k+1} + d_{2j}^{k+1} < (\lambda - 1)\rho_{4j}^{k+1} < \rho_{4j}^{k+1}$$

since $\lambda < 2$. This contradicts (ii).

The proof [2, p. 39; 3, p. 300] of the lemma fails where it is stated that ρ_{2j-1}^k cannot belong to both R'_{m-1} and R'_m. This is possible, the only requirement being that $\sigma_j^k = \rho_2^k \geq 2\pi\lambda^{1-m}$.

3. Theorem. *Let the perfect set P be constructed as described in* §1. *If $\epsilon_m = o(1)$ and $\theta_m = o(1/\eta_m)$ where $\eta_m = \sup_{n\geq m}(\epsilon_n)$, then P is a set of multiplicity for trigonometric series.*

Proof. Let

(1) $$\nu_m = \sup_{n\geq m} (\eta_n\theta_n), \qquad \text{so } \nu_m \to 0 \text{ monotonically.}$$

For $x \in P$ let $i_m(x) = i_m$ be determined by $x \in \rho_{i_m(x)}^m$. Then $\{\rho_{i_m(x)}^m\}$ is a unique sequence, strictly decreasing to zero. For $x \in P$ and a fixed n choose the unique $k = k(x, n)$ such that

(2) $$\frac{1}{(\nu_k)^{1/2}\rho_{i_k(x)}^k} \leq n < \frac{1}{(\nu_{k+1})^{1/2}\rho_{i_{k+1}(x)}^{k+1}}.$$

The set $\{\rho_{i_k(x)}^k\}$ covers P. This is a finite covering since every $x \in \rho_{i_{k+1}}^{k+1}$ determines the same $\rho_{i_k}^k$. By eliminating the superfluous elements we obtain a unique minimal cover V_n, consisting of nonoverlapping intervals.

Let ζ_n denote the minimum k such that $\rho_{i_k}^k \in V_n$. Then $\zeta_n \to \infty$ as $n \to \infty$.

Define

$$H_n(x) = \begin{cases} F_k(x) & \text{if } x \in \rho_{i_k}^k \in V_n, \\ F(x) & \text{if } x \notin V_n, \end{cases}$$

$$G_n(x) = \begin{cases} F_{k+1}(x) & \text{if} \quad x \in \rho_{i_k}^{k} \in V_n, \\ F(x) & \text{if} \quad x \notin V_n. \end{cases}$$

It is a standard result of the theory of uniqueness that the multiplicity will be established when we prove that

$$I_n = n \int_0^{2\pi} F(x) e^{-inx} dx = o(1). \tag{3}$$

Let T_1, T_2, and T_3 be defined by the relation

$$\begin{aligned} I_n &= n \int_0^{2\pi} [F(x) - G_n(x)] e^{-inx} dx \\ &\quad + n \int_0^{2\pi} [G_n(x) - H_n(x)] e^{-inx} dx \\ &\quad + n \int_0^{2\pi} H_n(x) e^{-inx} dx = T_1 + T_2 + T_3. \end{aligned} \tag{4}$$

Let $\sum_{(k)}$ denote the sum over all pairs (k, i_k) such that $\rho_{i_k}^k \in V_n$. Since $F(x) = F_{k+1}(x) = G_n(x)$ on $d_{i_k}^{k+1}$,

$$\begin{aligned} |T_1| &\leqq n \int_{V_n} |F - G_n| \, dx = n \sum_{(k)} \int_{\rho_{i_k}^k} |F - G_n| \, dx \\ &= n \sum_{(k)} \left[\int_{\rho_{2i_k-1}^{k+1}} |F - F_{k+1}| \, dx + \int_{\rho_{2i_k}^{k+1}} |F - F_{k+1}| \, dx \right]. \end{aligned}$$

In the construction of $F(x)$ it is shown [2, p. 32; 3, p. 293] that $|F(x) - F_m(x)| \leqq 4\eta_m |\Delta_{i_m}^m|$ on $\rho_{i_m}^m$, so

$$|T_1| \leqq \sum_{(k)} n 4 \eta_{k+1} \left[|\Delta_{2i_k-1}^{k+1}| \, \rho_{2i_k-1}^{k+1} + |\Delta_{2i_k}^{k+1}| \, \rho_{2i_k}^{k+1} \right].$$

From (2),

$$n < \frac{1}{(\nu_{k+1})^{1/2} \rho_{i_{k+1}}^{k+1}} \leqq \frac{1}{(\nu_{k+1})^{1/2} \tau_{i_k}^{k+1}}.$$

Thus

$$\begin{aligned} |T_1| &< \sum_{(k)} \frac{4\eta_{k+1}}{(\nu_{k+1})^{1/2}} \left[|\Delta_{2i_k-1}^{k+1}| \frac{\rho_{2i_k-1}^{k+1}}{\tau_{i_k}^{k+1}} + |\Delta_{2i_k}^{k+1}| \frac{\rho_{2i_k}^{k+1}}{\tau_{i_k}^{k+1}} \right] \\ &\leqq \sum_{(k)} \frac{4\eta_{k+1}\theta_{k+1}}{(\nu_{k+1})^{1/2}} \left[|\Delta_{2i_k-1}^{k+1}| + |\Delta_{2i_k}^{k+1}| \right]. \end{aligned}$$

Using (1),

$$(5) \qquad |T_1| < 4(\nu_{\zeta_n})^{1/2}\sum_{(k)}\left[|\Delta_{2i_k-1}^{k+1}| + |\Delta_{2i_k}^{k+1}|\right] = 8(\nu_{\zeta_n})^{1/2} = o(1).$$

By a partial integration, it follows that

$$T_2 = -i\int_0^{2\pi}(G_n' - H_n')e^{-inx}dx = -i\sum_{(k)}\int_{\rho_{i_k}^k}(F_{k+1}' - F_k')e^{-inx}dx.$$

Since $F_k = F_{k+1}$ on $\sigma_{i_k}^{k+1}$,

$$|T_2| \leqq \sum_{(k)}\left[\int_{\tau_{i_k}^{k+1}}|F_{k+1}' - F_k'|\,dx + \int_{d_{i_k}^{k+1}}|F_{k+1}' - F_k'|\,dx\right]$$

$$(6) \qquad = \sum_{(k)} 2\left[\text{absolute variation of } F_k \text{ on } d_{i_k}^{k+1}\right]$$

$$= 2\sum_{(k)}|\Delta_{i_k}^k|\frac{d_{i_k}^{k+1}}{\rho_{i_k}^k} \leqq 2\eta_{\zeta_n}\sum_{(k)}|\Delta_{i_k}^k| = o(1).$$

A partial integration in T_3 shows that

$$T_3 = -i\int_0^{2\pi}H_n'e^{-inx}dx = -i\sum_{(k)}\int_{\rho_{i_k}^k}F_k'e^{-inx}dx$$

$$= \sum_{(k)}\frac{\Delta_{i_k}^k}{\rho_{i_k}^k}\int_{\rho_{i_k}^k}e^{-inx}dx.$$

Finally, using (2),

$$|T_3| \leqq \sum_{(k)}\frac{2|\Delta_{i_k}^k|}{n\rho_{i_k}^k} \leqq 2\sum_{(k)}|\Delta_{i_k}^k|(\nu_k)^{1/2}$$

$$(7) \qquad \leqq 2(\nu_{\zeta_n})^{1/2}\sum_{(k)}|\Delta_{i_k}^k| = o(1).$$

Equations (4) through (7) imply (3), and the proof is complete.

References

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3. S. Verblunsky, *On a class of perfect sets*, Acta Math. vol. 65 (1935) pp. 291–305.

THE UNIVERSITY OF OREGON