MONOTONIC SUBSEQUENCES

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1. **Introduction.** Hidden in a paper by Erdös and Szekeres[1] is an intriguing result.

BASIC THEOREM. *Every sequence* $S=\{x_i\}$ $(i=1$ *to* $n^2+1)$ *of real numbers having* (n^2+1) *terms possesses a (perhaps not strictly) monotonic subsequence* $M=\{x_{i_j}\}$ $(j=1$ *to* $n+1)$ *having* $(n+1)$ *terms. Furthermore* (n^2+1) *is the smallest number for which this is true.*

Briefly, this theorem states that a monotonic subsequence of any desired length can be picked out from a sufficiently long sequence, and gives the precise lengths. An elegant proof of this theorem (unpublished) which is due to Martin D. Kruskal is sketched here.

NOTATION. Let S and T denote sequences, and let M and N denote monotonic sequences. Let $S(p)$, etc., denote a sequence having p terms. Let $\psi(n)$ denote the least integer p such that every $S(p)$ contains an $M(n)$. In this notation we may restate the basic theorem thus:

BASIC THEOREM. *For sequences of real numbers,* $\psi(n+1)=n^2+1$.

To show that $\psi(n+1)\geqq n^2+1$, it is sufficient to exhibit an $S(n^2)$ which contains no $M(n+1)$. Such a sequence is the following:

$$n, \cdots, 1, 2n, \cdots, n+1, \cdots, n^2, \cdots, n^2-n+1.$$

To show that $\psi(n+1)\leqq n^2+1$, assume the contrary and let n be the least integer such that $\psi(n+1)>n^2+1$. Let $S(n^2+1)$ be a sequence which does not contain any $M(n+1)$. Now define a *majorant* (*minorant*) of S to be a term which is strictly greater (smaller) than all terms following it in S. The majorants (minorants) form a decreasing (increasing) subsequence of S; hence there are at most n majorants and n minorants. As the final term of S is necessarily both a majorant and a minorant, there are at most $(2n-1)$ *extremants* (majorants and minorants). The last term of every $M(n)$ contained in S must be an extremant. Now delete from S all its extremants. The remainder S' can contain no $M(n)$, yet has at least $[(n^2+1)-(2n-1)]=[(n-1)^2+1]$ terms. Hence $\psi(n)\equiv\psi(n-1+1)>(n-1)^2+1$. This contradicts the definition of n and completes the proof.

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[1] P. Erdös and G. Szekeres, *A combinatorial theorem in geometry*, Compositio Mathematica vol. 2 (1935) p. 463. See the theorem on p. 467.

2. **First generalization—real vector spaces.** In the following two sections the concept of a monotonic sequence is generalized to sequences of vectors from a finite-dimensional real vector space and a partial analogue of the Basic Theorem is obtained. (If $S=\{x_i\}$ is a sequence from a vector space, the subscript still distinguishes terms of the sequence, *not* components of a vector.)

Note that a sequence $S=\{x_i\}$ of real numbers is monotonic if and only if all the differences $(x_{i+1}-x_i)$ lie (perhaps not strictly) to the same side of 0 on the real line. This motivates:

DEFINITION. A sequence $S=\{x_i\}$ of terms from a finite-dimensional real vector space is monotonic if there is a hyperplane H through the origin such that the differences $d_i \equiv x_{i+1}-x_i$ all lie in one of the closed half-spaces determined by H.

This definition is further justified by:

LEMMA 1. *A sequence S of vectors is monotonic if and only if there is a directed line L such that the perpendicular projections of the x_i on L form an increasing sequence.*

This lemma is easily proved by taking L and H to be perpendicular.

The direction of any such line L is called a direction of monotonicity of S. In 1-space there are only two possible directions of monotonicity: increasing and decreasing. In r-space the possible directions of monotonicity correspond to the points on the $(r-1)$-sphere.

Any sequence of two real numbers is monotonic. This generalizes to:

LEMMA 2. *In r-space any sequence of $(r+1)$ terms is monotonic.*

This can easily be proved geometrically.

NOTATION. For convenience, let the function $\psi(n)$ which applies in r-space be indicated by $\psi_r(n)$.

LEMMA 3. (a) *If $S(n^2+1)=\{x_i\}$ is any sequence of (n^2+1) terms in r-space and L is any directed line in r-space, then S contains either a subsequence monotonic in the direction of L or a subsequence monotonic in the direction opposite to L.* (b) $\psi_r(n+1)\leqq n^2+1$.

PROOF. (a) follows easily from Lemma 1 and the Basic Theorem; (b) follows from (a). But (a) is much stronger than (b) because (a) says "for any $L \cdots$" while (b) says implicitly "there exists an L such that. . . ." This suggests that actually $\psi_r(n+1)$ is smaller than (n^2+1) in general.

What is the full generalization of the Basic Theorem to r-space? In other words, what is the function $\psi_r(n)$?

CONJECTURE. $\psi_r(n+r)=rn+(n^2-n+1)$.

This conjecture is based solely on the following collection of facts.

BASIC THEOREM. $\psi_1(n+1)=n+(n^2-n+1)$.

LEMMA 2 (*New Form*). $\psi_r(1+r)=r+(1-1+1)$.

THEOREM 1. $\psi_r(2+r)\leqq 2r+3\equiv 2r+(4-2+1)$.

LEMMA 4. *For $r=1$ and 2, the $\leqq$ of Theorem 1 becomes $=$.*

The proof of Theorem 1 is long and occupies the next section. For $r=1$, Lemma 4 is trivial. To prove Lemma 4 for $r=2$, it is sufficient to exhibit a sequence of 6 vectors from 2-space which contains no monotonic subsequence of 4 terms. That the following is such a sequence may be verified directly:

$$(2, -1), (3, 6), (-3, 12), (-3, -12), (3, -6), (2, 1).$$

3. Proof of Theorem 1.

The basic tool in proving Theorem 1 is

LEMMA 5. *If $S(p)=\{x_i\}$ is a sequence in r-space, then at least one of the following conditions is true:*

(a) *S is monotonic;*

(b) *there exist real numbers $\alpha_i>0$ $(i=1, \cdots, p-1)$ such that $\sum\alpha_i d_i=0$, where $d_i\equiv x_{i+1}-x_i$.*

PROOF. It is sufficient to show that the falsity of (b) implies (a). Thus assume that 0 does not belong to the convex cone $C\equiv\{\sum\alpha_i d_i \mid \text{all } \alpha_i>0\}$. Then a well known theorem about convex sets yields that there is a hyperplane H through 0 such that $\overline{C}$ (the topological closure of C) lies entirely in one of the closed half-spaces determined by H. Since d_i is in $\overline{C}$ for all i, $S(p)$ is monotonic, and the proof is complete.

COMMENT. It is possible to modify (b) into a necessary and sufficient condition for non-monotonicity. This condition might be useful in further investigation of the function $\psi_r(n)$.

Lemma 6 follows from Lemma 5.

LEMMA 6. *Let*

$$S(p) = \{x_s\}$$

be any sequence in r-space, and let

$$S'(q) = \{x_{s_i}\}$$

be a non-monotonic subsequence of it (of course $1\leqq s_1<s_2<\cdots<s_q\leqq p$). Then there exists a $(p-1)$-tuple of real numbers $g=\{\gamma_s\}$ such

that $\sum \gamma_s d_s = 0$ *where* g *satisfies the following "suitability conditions with respect to* $(s_1, \cdots, s_q)$*":*

$$\begin{cases} \gamma_1 = \cdots = \gamma_{s_1-1} = 0, \\ \begin{cases} \gamma_{s_1} = \cdots = \gamma_{s_2-1} > 0, \\ \cdots\cdots\cdots\cdots, \\ \gamma_{s_{q-1}} = \cdots = \gamma_{s_q-1} > 0, \end{cases} \\ \gamma_{s_q} = \cdots = \gamma_{p-1} = 0. \end{cases}$$

PROOF. By Lemma 5 there exist strictly positive α_i $(i=1, \cdots, q-1)$ such that

$$\sum_1^{q-1} \alpha_i [x_{s_{i+1}} - x_{s_i}] = 0.$$

Hence

$$\sum_1^{q-1} \alpha_i \left[\sum_{s_i}^{s_{i+1}-1} d_j \right] = 0.$$

Now define $g = \{\gamma_s\}$ as follows:

$$\begin{cases} \gamma_1 = \cdots = \gamma_{s_1-1} = 0, \\ \begin{cases} \gamma_{s_1} = \cdots = \gamma_{s_2-1} = \alpha_1 > 0, \\ \cdots\cdots\cdots\cdots\cdots, \\ \gamma_{s_{q-1}} = \cdots = \gamma_{s_q-1} = \alpha_{q-1} > 0, \end{cases} \\ \gamma_{s_q} = \cdots = \gamma_{p-1} = 0. \end{cases}$$

Clearly $\sum \gamma_s d_s = 0$, and the proof is complete.

The structure of the $(p-1)$-tuple $g = (\gamma_1, \cdots, \gamma_{p-1})$ can be represented by a *q-block diagram*. This is obtained by substituting in g an "X" for each nonzero γ, and "o" for each zero γ, and an "$=$" for each comma between two γ's of one "block" of equal nonzero γ's. A g which is suitable with respect to (3, 5, 6, 9) and which has 10 components is represented by the following 10-dimensional 4-block diagram: (o, o, $X=X$, X, $X=X=X$, o, o). Block diagrams will be used extensively in the following arguments.

At this point it becomes necessary to consider the vectors of the fundamental r-space as r-tuples of real numbers. We shall write these r-tuples vertically and call them column vectors. We adopt the specific notation d_s = the column vector (δ_s^t) as $t=1, \cdots, r$, where d_s has its usual significance. The sequence $\{d_s\}$, with $s=1$ to

$p-1$, now becomes a matrix $D=\|\delta_s^t\|$ in which $t=1$ to r is the row index and $s=1$ to $p-1$ is the column index. We shall let d^t $(t=1, \cdots, r)$ represent the rows of D.

We now put Lemma 6 into the proper form for actual use:

LEMMA 7. *If $S(p)=\{x_s\}$ is a sequence in r-space, which does not contain any $M(q)$, then for each $(p-1)$-dimensional q-block diagram there exists a $(p-1)$-dimensional row vector g such that g is perpendicular to all the d^t $(t=1, \cdots, r)$ and such that g has the structure of the given q-block diagram.*

PROOF. The q-block diagram corresponds to a subsequence $S'(q)=\{x_{s_i}\}$ of S. Apply Lemma 6 to $S'(q)$ and rewrite the equation $\sum\gamma_s d_s=0$ as r equations $\sum\gamma_s\delta_s^t=0$. These may be written $g\cdot d^t=0$ or "g is perpendicular to d^t." This completes the proof.

The last tool needed to prove Theorem 1 is

LEMMA 8. *If $S(p)=\{x_s\}$ is not monotonic, the vectors d^t $(t=1$ to $r)$ are linearly independent.*

PROOF. As S is not monotonic, the d_s do not all lie on a common hyperplane through the origin, hence span the whole (r-dimensional) space of column vectors. Thus D has column-rank r, hence row-rank r, which completes the proof.

Theorem 1 is proved indirectly. Assume contrary to the theorem that $S(2r+3)=\{x_s\}$ contains no $M(r+2)$.

By Lemma 8, the r $(2r+2)$-dimensional row vectors d^t are linearly independent. Now apply Lemma 7 to the following $(r+2)$ different $(2r+2)$-dimensional $(r+2)$-block diagrams, and label the resulting g's as shown:

	1	2	$\cdots$	$r+1$	$r+2$	$r+3$	$\cdots$	$2r+2$
$g(1)$	$(X,$	$X,$	$\cdots,$	$X,$	o,	o,	$\cdots,$	o)
$g(2)$	(o,	$X,$	$\cdots,$	$X,$	$X,$	o,	$\cdots,$	o)
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$g(r+2)$	(o,	o,	$\cdots,$	o,	$X,$	$X,$	$\cdots,$	$X)$.

These g's are called the fundamental g's. Obviously the $(r+2)$ fundamental g's are linearly independent. By Lemma 7 every fundamental g is perpendicular to every d^t. Therefore $[g(1), \cdots, g(r+2), d^1, \cdots, d^r]$ is a basis for the $(2r+2)$-space of row vectors. From this follows

LEMMA 9. *Every g arising from application of Lemma 7 is a linear* *combination of the fundamental g's.*

At this point the proof of Theorem 1 splits into two cases, depending on whether r is odd or even; the former case is simpler and will be considered first.

Assume r is odd. Apply Lemma 7 to the following $(r+1)$ different $(2r+2)$-dimensional $(r+2)$-block diagrams and label the resulting g's as shown:

	1	2	$\cdots$	r	$r+1$		$r+2$	$r+3$	$r+4$	$\cdots$	$2r+2$
$\bar{g}(1)$	$(X,$	$X,$	$\cdots,$	$X,$	X	$=$	$X,$	o,	o,	$\cdots,$	o)
$\bar{g}(2)$	(o,	$X,$	$\cdots,$	$X,$	X	$=$	$X,$	$X,$	o,	$\cdots,$	o)
$\cdot\ \cdot\ \cdot\ \cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\bar{g}(r+1)$	(o,	o,	$\cdots,$	o,	X	$=$	$X,$	$X,$	$X,$	$\cdots,$	$X).$

With the aid of Lemma 9 it is easy to see that

$$\bar{g}(k) = \zeta(k)g(k) + \eta(k)g(k+1)$$

for properly chosen $\zeta(k)>0$ and $\eta(k)>0$. Introducing an obvious notation for the components of the g's and the $\bar{g}$'s, we have

$$\begin{aligned} \bar{\gamma}_{r+1}(k) &= \zeta(k)\gamma_{r+1}(k) + \eta(k)\gamma_{r+1}(k+1) \\ = \bar{\gamma}_{r+2}(k) &= \zeta(k)\gamma_{r+2}(k) + \eta(k)\gamma_{r+2}(k+1), \end{aligned}$$

which yields that

$$\left[\gamma_{r+1}(k) - \gamma_{r+2}(k)\right] = -\epsilon(k)\left[\gamma_{r+1}(k+1) - \gamma_{r+2}(k+1)\right]$$

where $\epsilon(k)$ is a positive constant. Now

$$\gamma_{r+1}(1) - \gamma_{r+2}(1) > 0$$

because the first term is positive and the second term is 0. The preceding equation now yields successively

$$\gamma_{r+1}(2) - \gamma_{r+2}(2) < 0, \qquad \gamma_{r+1}(3) - \gamma_{r+2}(3) > 0,$$

and so forth. Since r is odd, we obtain

$$\gamma_{r+1}(r+2) - \gamma_{r+2}(r+2) \equiv -\gamma_{r+2}(r+2) > 0$$

which is false. This completes the proof of Theorem 1 for odd values of r.

Now assume that r is even. The following notation is introduced for convenience:

$$\beta(k) \equiv \gamma_{r+1}(k) - \gamma_{r+2}(k), \qquad \beta'(k) \equiv \gamma_{r+1}(k) - \gamma_{r+3}(k).$$

Using the same method as in the preceding paragraph, the following inequalities are established (but without contradiction here):

$$\beta(k) > 0 \qquad \text{if } k \text{ is odd;}$$

$$\beta(k) < 0 \qquad \text{if } k \text{ is even.}$$

Now apply Lemma 7 to the following block diagrams and label the resulting g's as shown:

$$\begin{array}{c|l}
 & 1 \quad 2 \quad \cdots \quad r \quad r+1 \;\; r+2 \;\; r+3 \;\; r+4 \;\; r+5 \;\; \cdots \;\; 2r+2 \\
\hline
\bar{g}(1) & (X,\ X, \cdots, X, \quad X=X=X, \quad o, \quad o, \quad \cdots, \quad o) \\
\bar{g}(2) & (o,\ X, \cdots, X, \quad X=X=X, \quad X, \quad o, \quad \cdots, \quad o) \\
\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\bar{g}(r) & (o,\ o, \cdots, X, \quad X=X=X, \quad X, \quad X, \quad \cdots, \quad X).
\end{array}$$

With the aid of Lemma 9 and the established inequalities for $\beta(k)$, it is not difficult to show that

$$\bar{g}(k) = \zeta(k)g(k) + \eta(k)g(k+1) + \theta(k)g(k+2)$$

where $\zeta(k)$, $\eta(k)$, and $\theta(k)$ are positive constants. Translating these vector equations into component equations, and using the equalities among the components of the $\bar{g}$'s, we have

$$\begin{aligned}
&\bar{\gamma}_{r+1}(k) = \zeta(k)\gamma_{r+1}(k) + \eta(k)\gamma_{r+1}(k+1) + \theta(k)\gamma_{r+1}(k+2) \\
= \;&\bar{\gamma}_{r+2}(k) = \zeta(k)\gamma_{r+2}(k) + \eta(k)\gamma_{r+2}(k+1) + \theta(k)\gamma_{r+2}(k+2) \\
= \;&\bar{\gamma}_{r+3}(k) = \zeta(k)\gamma_{r+3}(k) + \eta(k)\gamma_{r+3}(k+1) + \theta(k)\gamma_{r+3}(k+2).
\end{aligned}$$

Subtract the second equation from the first, and then the third from the first:

$$0 = \zeta(k)\beta(k) + \eta(k)\beta(k+1) + \theta(k)\beta(k+2),$$

$$0 = \zeta(k)\beta'(k) + \eta(k)\beta'(k+1) + \theta(k)\beta'(k+2).$$

From these equations it follows that

$$\zeta(k) = \epsilon(k)\begin{vmatrix} \beta(k+1) & \beta(k+2) \\ \beta'(k+1) & \beta'(k+2) \end{vmatrix},$$

$$\eta(k) = \epsilon(k)\begin{vmatrix} \beta(k+2) & \beta(k) \\ \beta'(k+2) & \beta'(k) \end{vmatrix},$$

$$\theta(k) = \epsilon(k)\begin{vmatrix} \beta(k) & \beta(k+1) \\ \beta'(k) & \beta'(k+1) \end{vmatrix}$$

where $\epsilon(k)$ is a properly chosen constant of proportionality.

Call the three determinants $Z(k)$, $H(k)$, and $\Theta(k)$ respectively. Since $\zeta(k)$, $\eta(k)$, and $\theta(k)$ are all positive, $Z(k)$, $H(k)$, and $\Theta(k)$ must all have the same sign as $\epsilon(k)$. Furthermore, as $Z(k)\equiv\Theta(k+1)$, all the determinants have the same sign for all k (from 1 to r). To evaluate the sign of $\Theta(1)$ we use the already established inequalities for the β's and find the sign of the β''s from direct examination of the block diagrams of the g's. We see that

$$\Theta(1) = \begin{vmatrix} + & - \\ + & + \end{vmatrix} > 0,$$

so that all the determinants are positive.

Similarly, we find that

$$Z(1) = \begin{vmatrix} - & + \\ + & ? \end{vmatrix}.$$

For this to be positive, "?" must be "$-$", so that $\beta'(3)<0$. Using this result we see that

$$Z(2) = \begin{vmatrix} + & - \\ - & ? \end{vmatrix}.$$

For this to be positive, "?" must be "$+$," so that $\beta'(4)>0$. Similarly, $\beta'(5)<0$, $\beta'(6)>0$, and so forth. Since r is even, $\beta'(r+2)>0$; however, direct examination of the block diagram shows that $\beta'(r+2)<0$. This contradiction completes the proof of Theorem 1 for even values of r, and hence the whole proof.

4. Second generalization—relation spaces. In the following sections we again generalize the Basic Theorem, but in a manner quite different from that of the preceding sections.

The Basic Theorem is not in essence a statement about the real number system. To see this, consider any set X with an arbitrary binary relation $\subset$ over it. (No assumptions are made about $\subset$; for example, it need not be transitive.) Let us say that $S=\{x_i\}$ is $\subset$-monotonic ($\not\subset$-monotonic) if $x_i\subset x_{i+1}$ ($x_i\not\subset x_{i+1}$) for all i. Call S monotonic if it is either $\subset$-monotonic or $\not\subset$-monotonic. Then for sequences over X it is still true that $\psi(n+1)\leqq n^2+1$, and for a "general" space X it is true that $\psi(n+1)=n^2+1$. The inequality may be proved exactly as before.

What is the meaning of the "2" in (n^2+1)? The answer is simple: it is the number of relations ($\subset$ and $\not\subset$) of which at least one must hold between any two elements. The "2" is generalized to a "k" in Theorem 2.

DEFINITION. A k-relation space (kR-space for short) consists of a set X and k binary relations $\subset_h$ over X ($h=1, \cdots, k$) satisfying one axiom: for every x, y in X, there is at least one h depending on x and y such that $x \subset_h y$.

DEFINITION. A sequence $S=\{x_i\}$ is $\subset_h$-monotonic if $x_i \subset_h x_{i+1}$ for all i.

DEFINITION. A sequence S is monotonic if there is at least one h for which it is $\subset_h$-monotonic.

EXTENDED BASIC THEOREM. *For sequences over a 2-relation space, $\psi(n+1) \leqq n^2+1$. Furthermore, there are 2-relation spaces for which $\psi(n+1)=n^2+1$.*

THEOREM 2. *For sequences over a kR-space, $\psi(n+1) \leqq n^k+1$. Furthermore, there are kR-spaces for which $\psi(n+1)=n^k+1$.*

In the kR-space to be described $\psi(n+1)=n^k+1$. Let X consist of all real polynomials in the variable ξ of degree $\leqq k-1$. The relations $\subset_h$ are defined by

$$p(\xi) \subset_h q(\xi) \qquad \text{if } [p(\xi) - q(\xi)] \text{ has exactly degree } (k-h).$$

(The zero polynomial is assigned degree 0.) It is trivial to show that this is a kR-space, and the following $S(n^k)$ contains no $M(n+1)$:

$$\begin{array}{l}
\xi^{k-1} + \xi^{k-2} + \cdots + \xi + 1, \\
\xi^{k-1} + \xi^{k-2} + \cdots + \xi + 2, \\
\cdots\cdots\cdots\cdots\cdots, \\
\xi^{k-1} + \xi^{k-2} + \cdots + \xi + n, \\
\xi^{k-1} + \xi^{k-2} + \cdots + 2\xi + 1, \\
\cdots\cdots\cdots\cdots\cdots, \\
\xi^{k-1} + \xi^{k-2} + \cdots + 2\xi + n, \\
\cdots\cdots\cdots\cdots\cdots, \\
\cdots\cdots\cdots\cdots\cdots, \\
n\xi^{k-1} + n\xi^{k-2} + \cdots + n\xi + n.
\end{array}$$

The proof[2] that $\psi(n+1) \leqq n^k+1$ in a k-relation space rests on Lemma 10 which (for real numbers) is stated in the paper by Erdös and Szekeres.

[2] For the basic idea of this proof I am indebted to the referee, who suggested a proof far simpler than the one originally contained in my paper. However, by using Lemma 10, not originally in my paper and not known to the referee, I have further simplified his proof.

LEMMA 10. *If* $(X, \subset_1, \subset_2)$ *is a 2-relation space, then any sequence* $S(pq+1)$ *either contains a* $\subset_1$*-monotonic subsequence* $M(p+1)$ *or a* $\subset_2$*-monotonic subsequence* $M(q+1)$.

This lemma may easily be proved in the same way as the Extended Basic Theorem.

Now we proceed by an induction on k. If $(X, \subset_1, \cdots, \subset_{k+1})$ is a $(k+1)$-relation space, and $S(n^{k+1}+1)$ is a sequence over it, define $\ll_1$ and $\ll_2$ by

$$x \ll_1 y, \qquad \text{if } x \subset_h y \text{ for any } h \text{ from 1 to } k,$$
$$x \ll_2 y \qquad \text{if } x \subset_{k+1} y.$$

Now $(X, \ll_1, \ll_2)$ is a 2-relation space. Hence by Lemma 10, S contains either $M_1(n^k+1)$ which is $\ll_1$-monotonic or $M_2(n+1)$ which is $\ll_2$-monotonic. In the latter case the proof is complete as $M_2(n+1)$ is also $\subset_{k+1}$-monotonic. In the former case, let M_1 denote the set of elements in $M_1(n^k+1)$ and define $\subset^h$ over M_1 by

$$x \subset^h y \qquad \text{if } x \subset_h y \text{ or if } y \text{ precedes } x \text{ in } M_1(n^k + 1).$$

Then $(M_1, \subset^1, \cdots, \subset^k)$ is a kR-space. Hence by the induction hypothesis $M_1(n^k+1)$ must contain an $M(n+1)$ which is $\subset^h$-monotonic for some h from 1 to k. But then $M(n+1)$ is $\subset_h$-monotonic, which completes the proof.

5. De Bruijn's Theorem—a generalization.

In some unpublished work N. G. de Bruijn has generalized the Basic Theorem to sequences of m-tuples of real numbers. He defines such a sequence to be monotonic if each component sequence is monotonic. (Thus there are 2^m "directions" of monotonicity.)

DE BRUIJN'S THEOREM. *Over the space of m-tuples,* $\psi(n+1) = n^{2^m}+1$.

His proof is simply an m-fold application of the Basic Theorem. From $S(n^{2^m}+1)$ pick a subsequence $S_1(n^{2^{m-1}}+1)$ whose first components are monotonic. From S_1 pick a subsequence $S_2(n^{2^{m-2}}+1)$ whose second components are monotonic; and so forth. This eventually yields $S_m(n+1)$ which is monotonic. This shows that $\psi(n+1) \leq n^{2^m}+1$; the opposite inequality is easily verified.

De Bruijn's Theorem inspires Theorem 3, which is at once a generalization of De Bruijn's Theorem and of Theorem 2.

DEFINITION. A joint relation-space with coefficients $k_1, \cdots, k_m$ consists of m different relation-spaces over the same set X such that the lth space is a k_lR-space.

The relations are denoted by $\subset_h^l$, where $h=1, \cdots, k_l$ and $l=1, \cdots, m$.

DEFINITION. A sequence is monotonic over a joint relation-space if it is simultaneously monotonic over every one of the $k_l R$-spaces.

THEOREM 3. *Over a joint relation-space,* $\psi(n+1) \leqq n^{k_1 k_2 \cdots k_m}+1$.

De Bruijn's Theorem is a special case of Theorem 3 in which all the coefficients are 2 and the set X consists of the real m-tuples. However his proof cannot be extended to prove Theorem 3, for his proof depends on the transitivity of his relations which is not assumed in Theorem 3.

However Theorem 3 may be proved as a trivial corollary to Theorem 2. Simply define $k_1 k_2 \cdots k_m$ new relations over X by

$$x \ll_{h_1, \cdots, h_m} y \text{ only if } x \subset_{h_l}^l y \text{ for all } l.$$

Then X and the new relations form a $k_1 k_2 \cdots k_m R$-space. Use of Theorem 2 then completes the proof.

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