CONCERNING CONTINUOUS COLLECTIONS OF CONTINUOUS CURVES

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E. E. Moise has shown[1] that no compact irreducible continuum is filled up by a continuous collection of mutually exclusive arcs which is an arc with respect to its elements. The object of this paper is to present some extensions of this result. In the lemma and in Theorem 1, M will denote a compact metric continuum.

LEMMA. *If the domain D intersects a nondegenerate element z of a continuous collection G of mutually exclusive continuous curves which fills up a compact metric continuum M and is an arc with respect to its elements but $\overline{D}$ does not contain z, then for each element y of G distinct from z there is a connected subset of $\overline{D} \cdot M$ which intersects two elements of G distinct from y on the interval zy of elements of G.*

PROOF. Suppose the lemma is false. There exists an element a of G such that if b and c are elements of the interval za of G distinct from a, then no connected subset of $M \cdot \overline{D}$ intersects both b and c. Let P be a point of $z \cdot D$ and let R and D' be domains containing P such that $\overline{D}'$ is a subset of R and $\overline{R}$ is a subset of D. Since G is continuous, there exists, between z and a, an element x of G such that x, and every element of G between x and z, intersects both D' and $S-\overline{D}$. Let M' denote the sum of x, z, and all elements of G between them. Since no connected subset of $M' \cdot \overline{D}$ intersects both x and z, $M' \cdot \overline{D}$ is the sum of two mutually separated point sets K_1 and L_1 which contain $M' \cdot z$ and $M' \cdot x$ respectively.

Let y_1 denote the first element of G between z and x that intersects $L_1 \cdot \overline{D}'$. The element y_1 does not intersect $L_1 \cdot D'$, for, otherwise, the fact that G is continuous would imply the existence of an element of G *between* z and y_1 which intersects $L_1 \cdot D'$. However, y_1 does intersect D'; therefore it intersects $D' \cdot K_1$. But it also intersects $L_1 \cdot R$. Hence there is an element x_1 of G between z and y_1 such that x_1, and each element of G between it and y_1, intersects $K_1 \cdot D'$ and $L_1 \cdot R$, because both $K_1 \cdot D'$ and $L_1 \cdot R$ are open subsets of M.

No connected subset of K_1 intersects x_1 or some element of G preceding x_1 and also intersects y_1 or some element of G following y_1 in the order from z to x. Therefore K_1 is the sum of two mutually sepa-

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[1] E. E. Moise, *A Theorem on Monotone Interior Transformations*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 810–811.

rated point sets K_2 and L_2 such that K_2 contains the common part of K_1 and the sum of all the elements of the interval zx_1 of G and L_2 contains the common part of K_1 and the sum of all the elements of the interval y_1x.

Let y_2 denote the first element of G between z and x that intersects $L_2 \cdot \overline{D}'$. The element y_2 precedes y_1 in the order from z to x and therefore does not intersect $L_1 \cdot D'$. Since G is continuous y_2 does not intersect $L_2 \cdot D'$. Consequently it intersects $K_2 \cdot D'$. Since it intersects L_2 it follows x_1 in the order from z to x and therefore intersects $L_1 \cdot R$.

The elements chosen are in the order z, x_1, y_2, y_1, x. Since y_2 intersects $L_2 \cdot R$, $L_1 \cdot R$, and $K_2 \cdot D'$, there is, between x_1 and y_2, an element x_2 of G such that every element of the interval x_2y_2 of G intersects $L_2 \cdot R$, $L_1 \cdot R$, and $K_2 \cdot D'$. No connected subset of K_2 intersects x_2 or some element of G preceding it and also intersects y_2 or some element following it. Therefore K_2 is the sum of two mutually separated point sets K_3 and L_3 such that K_3 contains the common part of K_2 and the sum of all the elements of the interval zx_2 and L_3 contains the common part of K_2 and the sum of all the elements of y_2x.

Let y_3 be the first element of G between z and x that intersects $L_3 \cdot \overline{D}'$. The element y_3 precedes y_2 and follows x_2. Therefore, as may be seen with the aid of arguments similar to the preceding, y_3 intersects $K_3 \cdot D'$, $L_3 \cdot R$, $L_2 \cdot R$, and $L_1 \cdot R$. The elements chosen are in the order $z, x_1, x_2, y_3, y_2, y_1, x$.

Continuing in this manner we obtain sequences $L_1, L_2, L_3, \cdots$, $K_1, K_2, K_3, \cdots$, and $y_1, y_2, y_3, \cdots$ such that for each positive integer n, (1) K_n is the sum of K_{n+1} and L_{n+1} where K_{n+1} and L_{n+1} are mutually separated, (2) K_n+L_n is a subset of D, (3) y_{n+1} precedes y_n and follows x_1 in the order from z to x, and (4) if n is not less than i, y_n intersects $L_i \cdot R$. The sequence $y_1, y_2, y_3, \cdots$ has a sequential limiting set, y, which is an element of G between z and x. Since for each n infinitely many elements of the sequence $y_1, y_2, y_3, \cdots$ intersect $L_n \cdot \overline{R}$, y intersects $L_n \cdot \overline{R}$.

For each positive integer i let Q_i denote a point of the common part of y and $L_i \cdot \overline{R}$ and let Q be a limit point of the sequence $Q_1, Q_2, Q_3, \cdots$. Since, for each n, K_n and L_n are mutually separated and L_i is a subset of K_n if i is greater than n, Q is not a point of L_n. Therefore it is a point of K_n for each n. Since y is a continuous curve there exists a domain h containing Q such that $\bar{h}$ is a subset of D and $y \cdot h$ is connected. The point set $y \cdot h$ is a domain with respect to y. Therefore there exists a positive integer j such that Q_j is a point of $y \cdot h$. However, Q_j is a point of L_j and Q is a point of K_j, so $y \cdot h$ is a connected subset of M' intersecting K_j and L_j. This is impossible since M' is the sum of the mutually separated point sets K_j, L_j, L_{j-1}, $\cdots$ and L_1. This involves a contradiction. Hence the lemma is proved.

THEOREM 1. *If a continuous collection G of mutually exclusive continuous curves, not all degenerate, is an arc with respect to its elements, then the sum of all the elements of G is not irreducible.*

PROOF. Let a and b denote the end elements, and let M denote the sum of all the elements, of G, whence M is a compact metric continuum. If P and Q are points of the same element of G, M is not irreducible between P and Q. If P is a point of the element x and Q of the element y and either x or y is not an end element of G, the sum of x, y and all elements of G between them is a proper subcontinuum of M and therefore M is not irreducible between P and Q.

Suppose P is a point of a and Q is a point of b. Let z denote a nondegenerate element of G distinct from a and b and let D be a domain intersecting z but such that z is not a subset of $\overline{D}$. Let y be an element of G between z and b such that every element other than z of the interval zy intersects D and $S-\overline{D}$. It follows from the lemma, however, that there are elements x and x' of G between z and y such that x' is between x and y and that there is a continuum K such that K is a subset of $\overline{D}\cdot M$ and intersects both x and x'. Let L be the sum of K, x, and all elements of G preceding x in the order from a to b together with x' and all elements following it in the order from a to b. The point set L is a proper subcontinuum of M and contains P and Q. Therefore M is not irreducible.

THEOREM 2. *If the continuous collection G of mutually exclusive arcs fills up the compact continuum M in the plane and is an arc with respect to its elements, then M is a domain plus its boundary.*

PROOF. Suppose the domain D contains the point P of the element z of G. Let J denote a simple closed curve whose interior I contains P such that $J+I$ lies in D and does not contain z. Let U be a domain containing P such that $\overline{U}$ is a subset of I. Let y be an element of G such that every element of the interval yz of G contains a point of U and a point not in $J+I$. There are elements x and x' of G between z and y and a continuum K such that K is a subset of $M\cdot\overline{U}$ intersecting x and x'.

Let Q be a point of $x\cdot I$ not in $\overline{U}$. Let V be a domain containing Q such that $\overline{V}$ is a subset of $I-\overline{U}$. Let y' be an element of G between x and x' such that every element of the interval xy' of G except x contains a point of V and a point not in $J+I$. There are elements w

and w' of G between x and y' and a continuum K' such that K' is a subset of $M \cdot \overline{V}$ intersecting both w and w'.

Since they are between x and x', w and w' intersect K. Since $\overline{U}$ and $\overline{V}$ are mutually separated, the common part of $K+w$ and $K'+w'$ is not connected and $K+w+K'+w'$ is a continuum, L, which separates the plane. The point set $S-L$ is the sum of two mutually separated point sets R and T such that T is unbounded and connected. However, since no element of G separates the plane and G is an arc with respect to its elements, $S-M$ is connected and is consequently a subset of T. Hence R is a subset of M. Let W be a component of R. The point set W is a complementary domain of L and its boundary, B, separates S. The continuum B is a subset of L and, since no subset of $w+w'$ separates S, B contains a point of $K+K'$ not in $w+w'$. There is, consequently, a point of W in $D \cdot M$ and $D \cdot M$ contains a domain.

We have shown that every domain which intersects M contains a domain which is a subset of M. This is sufficient to show that M is a domain plus its boundary.

THEOREM 3. *Under the hypotheses of Theorem* 2, *if M is a continuous curve it is a simple closed curve plus its interior.*

PROOF. Let B denote the boundary of M. Since $S-M$ is connected, B is connected. It has no cut point, for if P is a point of B and $B-P$ is not connected, $M-P$ is the sum of two mutually separated point sets K and L. Let z denote the element of G containing P. Every other element of G is a subset either of L or of K. From this and the fact that G is a continuous collection it follows that z does not intersect K or L. This contradicts the fact that z is an arc. Hence B has no cut point. But it is a continuous curve and is the boundary of a domain. It follows that it is a simple closed curve.

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