CONTINUA WHICH ARE THE SUM OF A FINITE NUMBER OF INDECOMPOSABLE CONTINUA

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Swingle [7][1] has given the following definitions. (1) A continuum M is said to be the *finished sum* of the continua of a collection G if $G^* = M$ and no continuum of G is a subset of the sum of the others.[2] (2) If n is a positive integer, the continuum M is said to be *indecomposable under index n* if M is the finished sum of n continua and is not the finished sum of $n+1$ continua.

Swingle has shown [7, Theorem 2] that if n is a positive integer and the continuum M is indecomposable under index n, then M is the finished sum of n indecomposable continua. The author has shown [2, Theorem 1] that if $n=2$ and the continuum M is indecomposable under index n, and G is a collection of n indecomposable continua whose finished sum is M, then G is the only such collection. In the present paper, it is shown that for a compact continuum, this theorem holds for any positive integer n. Also, there is given a necessary and sufficient condition that a compact continuum be indecomposable under index n.

An indecomposable continuum can be described as a nondegenerate continuum which is indecomposable under index 1. If $n=1$, then in order that a continuum M be indecomposable under index n, it is necessary and sufficient that M contain $n+2$ points such that M is irreducible about any $n+1$ of them.[3] Swingle [7] has shown that it is impossible, in a certain manner, to generalize this theorem. Theorem 3 of the present paper might be considered a generalization of the necessary condition of the above theorem. However, it is easily seen that the converse of Theorem 3 is not true.

Theorems 1–5 are proved on the basis of R. L. Moore's Axioms 0 and 1_3. Hence these theorems hold in any metric space.[4]

THEOREM 1. *If $n>1$ and the compact continuum M is the sum of n indecomposable continua $M_1, M_2, \cdots, M_n$ such that, for each i ($i \leqq n$), a composant[5] K_i of M_i does not intersect $M_1+M_2+\cdots+M_{i-1}$*

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

[2] The sum of the continua of G is denoted by G^*.

[3] For a proof of this theorem, see [4, Theorem IV].

[4] Moore's axioms are stated in [5]. The first three parts of Axiom 1 are denoted by Axiom 1_3.

[5] If P is a point of a continuum M, the set of all points X such that $P+X$ lies in a proper subcontinuum of M is called a composant of M.

$+M_{i+1}+\cdots+M_n$, *then* M *is indecomposable under index* n.

PROOF. Suppose that there is a collection G consisting of $n+1$ continua whose finished sum is M. No continuum of G is a proper subset of one of the indecomposable continua $M_1, M_2, \cdots, M_n$. Hence, for each i ($i\leqq n$), if K_i intersects a continuum X of G, then X contains M_i. Consequently, there exist n continua of G such that their sum is M. This is contrary to the supposition that M is the finished sum of the continua of G. Since M is the finished sum of the continua $M_1, M_2, \cdots, M_n$, then it is indecomposable under index n.

THEOREM 2. *If* n *is a positive integer and the compact continuum* M *is indecomposable under index* n, *then there is only one collection of indecomposable continua whose finished sum is* M.

PROOF. By [7, Theorem 2], there is a collection G consisting of n indecomposable continua $M_1, M_2, \cdots, M_n$ such that M is their finished sum. By [3, Theorem 1], for each i ($i\leqq n$), some composant K_i of M_i does not intersect $(G-M_i)^*$. Suppose that there is a collection G' of indecomposable continua such that $G'\neq G$ and M is the finished sum of the continua of G'. Let i be a positive integer not greater than n. Some continuum X_i of G' intersects K_i. Neither of the indecomposable continua X_i and M_i is a proper subset of the other. Since no proper subcontinuum of M_i intersects both K_i and $(G-M_i)^*$, then $X_i=M_i$. Hence $G'=G$.

THEOREM 3. *If* $n>1$ *and the compact continuum* M *is indecomposable under index* n, *then there is a subset* H *of* M *consisting of* $2n$ *points such that* M *is irreducible about every subset of* H *consisting of* $2n-1$ *points.*

PROOF. Let $M_1, M_2, \cdots, M_n$ be n indecomposable continua whose finished sum is M. For each i ($i\leqq n$), let K_i be a composant of M_i as described in the proof of Theorem 2. There exists a subset H of M such that for each i ($i\leqq n$), $H\cdot M_i$ consists of two points of K_i. The set H satisfies the requirements of the conclusion of Theorem 3.

THEOREM 4. *If* $n>1$, M *is a compact continuum,* G *is a collection consisting of* n *indecomposable continua whose finished sum is* M, *and* H *is a finite set of points about which* M *is irreducible, then* M *is indecomposable under index* n.

LEMMA 4.1. *If the hypothesis of Theorem* 4 *is satisfied,* X *is a continuum of* G, *and* T *is a component of* $(G-X)^*$, *then some composant of* X *does not intersect* T.

Proof of Lemma 4.1. Suppose that every composant of X intersects T. Then there exists a finite collection W of proper subcontinua of X such that $W^*+(G-X)^*$ is connected. There exists a finite collection Y of proper subcontinua of X such that (1) every continuum of Y intersects $(G-X)^*$ and (2) if X intersects H, then Y^* contains $X\cdot H$. Since X is indecomposable and M is the finished sum of the continua of G, then Y^*+W^* does not contain[6] $M-(G-X)^*$. Therefore, $W^*+Y^*+(G-X)^*$ is a proper subcontinuum of M containing H. This is a contradiction since M is irreducible about H.

Proof of Theorem 4. An inductive argument will be used. Suppose that Theorem 4 is not true. Let k be the smallest positive integer n such that if M is a compact continuum satisfying the hypothesis of Theorem 4, then M is not indecomposable under index n. By Theorem 1, there is a continuum X of G such that every composant of X intersects $(G-X)^*$. By Lemma 4.1, $(G-X)^*$ is not connected. Therefore, $k>2$. The set $(G-X)^*$ is the sum of a finite number of mutually exclusive continua. Let T be one of these continua. Since M is irreducible about H, then $T-T\cdot X$ contains a point of H. By Lemma 4.1, there is a composant of X which does not intersect T. Let P be a point of such a composant. The continuum $T+X$ is irreducible about the finite set $H\cdot T+P$. There is a positive integer j less than k such that $T+X$ is the finished sum of j continua of G. Then $T+X$ is indecomposable under index j. By [3, Theorem 1], every continuum of G which is a subset of $T+X$ contains a composant which does not intersect any other continuum of G which is a subset of $T+X$. Therefore, every continuum of $G-X$ contains a composant which does not intersect any other continuum of G. Let L be a collection consisting of $k-1$ points such that if Z is a continuum of $G-X$, then a point of L belongs to a composant of Z lying in $M-(G-Z)^*$. Since, by supposition, M is not indecomposable under index k, then there is a collection G' consisting of $k+1$ continua whose finished sum is M. Since the set L is contained in the sum of $k-1$ continua of G', then $(G-X)^*$ is contained in the sum of $k-1$ continua of G'. Hence there exist two continua X_1 and X_2 of G' such that each of them contains a point of $M-(G-X)^*$ which does not belong to any other continuum of G'. Let R be a domain intersecting X_1 and not intersecting $(G'-X_1)^*+(G-X)^*$. Every composant of X intersects R. Therefore, there exists a finite collection W of proper sub-

[6] This follows from the fact that every proper subcontinuum of an indecomposable continuum M is a continuum of condensation of M [4, Theorem II] and the fact that no indecomposable continuum is the sum of a finite number of its proper subcontinua [4, Theorem III].

continua of X such that $X_1+W^*+(G-X)^*$ is a continuum. Let Y be a finite collection of continua as described in the proof of Lemma 4.1. Since $X_1+Y^*+W^*+(G-X)^*$ is a subcontinuum of M containing H, then $X_1+Y^*+W^*+(G-X)^*=M$. Since X is indecomposable and $X_1+(G-X)^*$ contains $X-(Y^*+W^*)$, then $X_1+(G-X)^*$ contains X. This is impossible since $X_1+(G-X)^*$ does not contain X_2. Thus the supposition that Theorem 4 is not true has led to a contradiction.

THEOREM 5. *If $n>1$, then in order that the compact continuum M should be indecomposable under index n, it is necessary and sufficient that M should be the finished sum of n indecomposable continua and be irreducible about some n points.*[7]

The necessity follows from [7, Theorem 2] and [3, Theorem 2]. The sufficiency follows from Theorem 4.

THEOREM 6. *If the compact continuum M in the plane is the finished sum of two indecomposable continua H and K such that some composant of H does not intersect K, then M is indecomposable under index two.*

LEMMA 6.1. *If the hypothesis of Theorem 6 is satisfied and K_1 and K_2 are mutually exclusive simple discs*[8] *intersecting K but not H, then there do not exist four mutually exclusive continua W_1, W_2, W_3, and W_4 such that, for each i $(i\leqq 4)$, W_i belongs to K, intersects H, and is irreducible from K_1 to K_2.*

PROOF OF LEMMA 6.1. Suppose that there do exist four such continua. Let D denote the complementary domain of K_1+K_2. Consider the case in which W_3+W_4 separates W_1 from W_2 in $\overline{D}$. Let R_1 and R_2 be connected domains intersecting $H\cdot W_1$ and $H\cdot W_2$ respectively and not intersecting $K_1+K_2+W_3+W_4$. There is a composant L of H which intersects both R_1 and R_2 and lies in $M-K$. Then L intersects $K_1+K_2+W_3+W_4$. This is a contradiction since $M-K$ does not intersect $K_1+K_2+W_3+W_4$.

PROOF OF THEOREM 6. Suppose, on the contrary, that M is the finished sum of three continua M_1, M_2, and M_3. One of these three continua intersects a composant of H lying in $M-K$. Suppose that M_1 is such a continuum. Then it contains H and intersects each of

[7] For an example showing that this theorem does not hold true without the condition that M be irreducible about some n points, see [1, p. 540]. Also, see [2, Example 1]. Sorgenfrey [6] has proved a theorem giving a necessary and sufficient condition that a compact continuum be irreducible about some n points.

[8] In the plane, a simple closed curve together with its interior is called a simple disc.

the continua M_2 and M_3. Each of the continua M_3 and M_1+M_2 contains a point of K not belonging to the other of these two continua. Since the closure of $M-(M_1+M_2)$ is a proper subset of the indecomposable continuum K, then $M-(M_1+M_2)$ is not connected. Let T_1 and T_2 be two mutually separated sets whose sum is $M-(M_1+M_2)$. Let K_1 and K_2 be two mutually exclusive simple discs whose interiors intersect T_1 and T_2 respectively such that K_1 and K_2 do not intersect $T_2+M_1+M_2$ and $T_1+M_1+M_2$ respectively. Since every composant of K intersects both K_1 and K_2, there exist six distinct composants of K each of which contains a continuum irreducible from K_1 to K_2. By Lemma 6.1, at most three of these intersect H, and hence three do not. Denote three which do not by W_1, W_2, and W_3. Let D denote the complementary domain of K_1+K_2. There exist two of the continua W_1, W_2, and W_3 such that their sum separates the other one from H in $\overline{D}$. Consider the case in which W_1+W_3 separates W_2 from H in $\overline{D}$. Let I denote the complementary domain of $K_1+K_2+W_1+W_3$ which contains the connected set $W_2-W_2\cdot(K_1+K_2)$. Since one of the sets $K_1\cdot W_2$ and $K_2\cdot W_2$ belongs to T_1 and the other to T_2, then $I\cdot W_2$ contains a point of the continuum M_1+M_2. Since H is a subset of M_1+M_2 and does not intersect $\overline{I}$, then there is a continuum Z belonging to $\overline{I}\cdot(M_1+M_2)$ and intersecting both W_2 and W_1+W_3. But this is impossible since Z is a proper subcontinuum of K intersecting two composants of K. Thus the supposition that M is the finished sum of three continua has led to a contradiction.

THEOREM 7. *If the hypothesis of Theorem* 6 *is satisfied, then uncountably many composants of K lie in $M-H$.*

This theorem follows from Theorem 6 and [3, Theorem 1].

REMARK. *Neither Theorem* 6 *nor Theorem* 7 *holds true in Euclidean three-dimensional space.* Let H' be the point set obtained by translating the point set H of [2, Example 1] one-half unit to the left. Let H'' be a point set obtained by revolving H' through 90 degrees about the vertical line whose equation is $x=1/2$. Only one composant of H'' intersects H, but every composant of H intersects H''. It follows from [3, Theorem 1] and Theorem 2 that the continuum $H+H''$ is not indecomposable under index two.

Added in proof. I have recently observed that Theorem 6 follows from Theorem 1 and a lemma proved by N. E. Rutt [*Some theorems on triodic continua*, Amer. J. Math. vol. 56 (1934) pp. 122–132 Lemma I]. I regret that I was not aware of Rutt's lemma at the time I prepared this paper.

BIBLIOGRAPHY

1. Paul Alexandroff, *Über kombinatorische Eigenschaften allgemeiner Kurven*, Math. Ann. vol. 96 (1927) pp. 512–554.

2. C. E. Burgess, *Continua and their complementary domains in the plane*, Duke Math. J. vol. 18 (1951) pp. 901–917.

3. ———, *Continua and their complementary domains in the plane*. II, Duke Math. J. vol. 19 (1952) pp. 223–230.

4. S. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fund. Math. vol. 1 (1920) pp. 210–222.

5. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications, vol. 13, 1932.

6. R. H. Sorgenfrey, *Concerning continua irreducible about n points*, Amer. J. Math. vol. 68 (1946) pp. 667–671.

7. P. M. Swingle, *Generalized indecomposable continua*, Amer. J. Math. vol. 52 (1930) pp. 647–658.

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