NOTE ON S-NUMBERS

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The transcendental number ξ is said to be an S-number if there is a $\gamma>0$ and a sequence of positive constants $\Gamma_1, \Gamma_2, \cdots$ such that for each polynomial $a_0+\cdots+a_m x^m$ of arbitrary degree m, with rational integral coefficients such that $a=\max(|a_0|, \cdots, |a_m|)\geqq 1$, the inequality

$$(1) \qquad |a_0+a_1\xi+\cdots+a_m\xi^m| \geqq \Gamma_m a^{-\gamma m}$$

holds. It is well known (cf. [1, p. 63]) that if ξ is an S-number, then $\gamma \geqq 1/\sigma$, where

$$\sigma = \begin{cases} 1 \text{ if } \xi \text{ is real,} \\ 2 \text{ if } \xi \text{ is not real,} \end{cases}$$

and Khintchine [2] has shown that the measure of all real S-numbers with $\gamma=1$ is zero. On the other hand, Mahler [3] proved that almost all (real or complex) numbers are S-numbers. His proof shows that this statement is true with $\gamma=4$, and he conjectured that for almost all real numbers one can take $\gamma=1+\epsilon$, and that for almost all complex numbers one can take $\gamma=1/2+\epsilon$, for arbitrary $\epsilon>0$.

Let γ_r be the infimum of all numbers γ' such that almost all real numbers are S-numbers with $\gamma\leqq\gamma'$, and define γ_c analogously in the complex case; thus Mahler's conjecture is that $\gamma_r=1$, $\gamma_c=1/2$. Koksma [4] showed that $\gamma_r\leqq 3$ and $\gamma_c\leqq 5/2$. Kubilyus [5] considered the special case $m=2$, and showed that then (1) holds for almost all ξ if $\gamma>1/\sigma$; i.e., he proved the conjecture for $m=2$. (It is well known for $m=1$.) By combining a theorem due to Fel'dman [6] with Mahler's original argument, it is shown here that $\gamma_r\leqq 2$, $\gamma_c\leqq 3/2$.

Lemma 5 of Fel'dman's paper is as follows: Let $f(z)=a_0+\cdots+a_m z^m$, where $a_0, \cdots, a_m$ are rational integers with $|a_i|\leqq a$, let $\xi_1, \cdots, \xi_m$ be its zeros, which are supposed distinct, and let ζ be an arbitrary complex number. If $\delta=\min_i(|\zeta-\xi_i|)$, then

$$|f(\zeta)| \geqq \delta e^{-2m^2}a^{-m}.$$

First consider the real case. If f satisfies the conditions of Fel'dman's lemma and

$$(2) \qquad |f(x)| \leqq \mu a^{-2m-3},$$

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then

$$\delta \leqq \mu e^{2m^2} a^{-m-3}.$$

Thus we have the following analogue of Lemma 2 of [**3**]: The point set $M(f)$ on the real axis, in which the polynomial $f(x)$ satisfies the inequality (2), has total length at most

$$2m\mu e^{2m^2} a^{-m-3}.$$

The remaining argument of [**3**] can be applied almost unchanged to this variant of Lemma 2, and the inequality $\gamma_r \leqq 2$ results. (This constant 2 is the coefficient of m in (2).)

In the complex case, if

$$|f(z)| \leqq \mu a^{-3(m+1)/2},$$

then

$$\delta \leqq \mu e^{2m^2} a^{-(m+3)/2},$$

and so z must lie in a set composed of at most m circular regions, of total area at most

$$\pi m \mu^2 e^{4m^2} a^{-m-3},$$

and Mahler's argument leads to the inequality $\gamma_c \leqq 3/2$.

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