A NOTE ON BERNOULLI AND EULER NUMBERS OF ORDER $\pm p$

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1. **Introduction.** The Bernoulli numbers of order k may be defined by [6, p. 143]

$$\left(\frac{x}{e^x - 1}\right)^k = \sum_{m=0}^{\infty} \frac{x^m}{m!} B_m^{(k)} \qquad (B_m = B_m^{(1)}). \tag{1.1}$$

For $k=p$, put

$$A_r = (-1)^r \binom{p-1}{r} B_r^{(p)} \qquad (0 \leqq r \leqq p-1); \tag{1.2}$$

then as is well known

$$(x+1)(x+2)\cdots(x+p-1) = x^{p-1} + A_1 x^{p-2} + \cdots + A_{p-1}.$$

Glaisher [4, p. 325] has established the congruences

$$\begin{aligned} A_{2r} &\equiv -\frac{1}{2r} pB_{2r} \pmod{p^2}, \\ A_{2r+1} &\equiv \frac{2r+1}{4r} p^2 B_{2r} \pmod{p^3} \end{aligned} \tag{1.3}$$

for $1 \leqq r \leqq (p-3)/2$, p prime >3. On the other hand the writer [1] has proved that

$$B_p^{(p)} \equiv p^2/2 \pmod{p^3} \tag{1.4}$$

and indeed [2] the more precise result

$$B_p^{(p)} \equiv -p^2(p-1)!/2 \pmod{p^5}. \tag{1.5}$$

Now for $k=-p$, Nielsen [5, p. 338] has proved that

$$\begin{aligned} B_{2r}^{(-p)} &\equiv \frac{1}{2r} pB_{2r} \pmod{p^2}, \\ B_{2r+1}^{(-p)} &\equiv \frac{2r+1}{4r} p^2 B_{2r} \pmod{p^3}, \end{aligned} \tag{1.6}$$

where $1 \leqq r \leqq (p-3)/2$.

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In the next place we define the numbers $C_m^{(k)}$ by means of [6, p. 143]

$$\left(\frac{2}{e^x+1}\right)^k = \sum_{m=0}^{\infty} \frac{C_m^{(k)}}{2^m} \frac{x^m}{m!} \qquad (C_m = C_m^{(1)}),$$

so that the $C_m^{(k)}$ are closely related to the Euler numbers of order k. Corresponding to (1.3) and (1.6) we have, for $k=p$,

$$\begin{aligned} C_{2r}^{(p)} &\equiv pC_{2r-1} \pmod{p^2}, \\ C_{2r+1}^{(p)} &\equiv -(2r+1)p^2C_{2r-1} \pmod{p^3}, \end{aligned} \tag{1.7}$$

where now $r \geqq 1$. For $k=-p$, we have

$$\begin{aligned} C_{2r}^{(-p)} &\equiv -pC_{2r-1} \pmod{p^2}, \\ C_{2r+1}^{(-p)} &= -(2r+1)p^2C_{2r-1} \pmod{p^3} \end{aligned} \tag{1.8}$$

for $r \geqq 1$; (1.8) is due to Nielsen [5, p. 292]. We remark that (1.3), (1.6), (1.7), (1.8) are proved in a uniform manner in [3].

In view of the above it is natural to seek congruences corresponding to (1.5) for the numbers $B_p^{(-p)}$, $C_p^{(p)}$, $C_p^{(-p)}$. Since

$$B_m^{(-k)} = \frac{m!}{(m+k)!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} r^{m+k},$$

it seems convenient, in analogy with (1.2), to define

$$\overline{A}_m = \frac{1}{p!} \sum_{r=0}^{p} (-1)^{p-r} \binom{p}{r} r^{m+p}. \tag{1.9}$$

Then we shall show that

$$\overline{A}_p \equiv p^2\overline{A}_{p-1} + \frac{1}{18} p^5 B_{p-3} \pmod{p^6}, \tag{1.10}$$

while

$$\overline{A}_{p-1} \equiv \frac{2pB_{p-1}}{p-1} + (p-1)! \pmod{p^3}. \tag{1.11}$$

In particular (1.10) and (1.11) imply

$$\overline{A}_p \equiv p^2(p-1)! + \frac{2p^3B_{p-1}}{p-1} \pmod{p^5}. \tag{1.12}$$

In the next place for $C_p^{(p)}$ we prove

(1.13) $$C_p^{(p)} \equiv -p^2(p-1)C_{p-2} \pmod{p^4}.$$

For $C_p^{(-p)}$ we prove

(1.14) $$C_p^{(-p)} \equiv p^2 C_{p-1}^{(-p)} + \tfrac{2}{3}p^5 C_{p-4} \pmod{p^6};$$

also

(1.15) $$C_{p-1}^{(-p)} + pC_{p-2} \equiv C_{p-1}^{(p)} \equiv p^2 \sum_1^{p-4} \binom{p-2}{s} C_s C_{p-3-s} \pmod{p^3}.$$

We suppose throughout that p is a prime >3.

2. **Proof of (1.10) and (1.11).** To prove (1.10) we make use of the following formula [5, p. 293]:

(2.1) $$\overline{A}_r = \sum_{s=0}^{r} (-1)^s \binom{p+r}{p+s} p^{r-s} \overline{A}_s,$$

which can be proved without much trouble using (1.9). Now for $r=p$, (2.1) becomes

(2.2) $$\begin{aligned} 2\overline{A}_p &= \sum_{s=1}^{p} (-1)^{p-s} \binom{2p}{s} p^s \overline{A}_{p-s} \\ &\equiv 2p^2\overline{A}_{p-1} - \binom{2p}{2} p^2 \overline{A}_{p-2} + \binom{2p}{3} p^3 \overline{A}_{p-3} \pmod{p^6}. \end{aligned}$$

But by (1.6) we have

$$\overline{A}_{p-2} \equiv \tfrac{1}{3}p^2 B_{p-3} \pmod{p^3}, \qquad \overline{A}_{p-3} \equiv -\tfrac{1}{3}pB_{p-3} \pmod{p^2},$$

so that (2.2) yields

$$\begin{aligned} 2\overline{A}_p &\equiv 2p^2\overline{A}_{p-1} + \tfrac{1}{3}p^5 B_{p-3} - \tfrac{4}{9}p^5 B_{p-3} \\ &\equiv 2p^2\overline{A}_{p-1} + \tfrac{1}{9}p^5 B_{p-3} \pmod{p^6}, \end{aligned}$$

which is equivalent to (1.10).

To prove (1.11) we use the formula [6, p. 146]

(2.3) $$B_m^{(k)} = -\frac{k}{m} \sum_{s=1}^{m} (-1)^s \binom{m}{s} B_s B_{m-s}^{(k)}.$$

In (2.3) take $m=p-1$, $k=p$ and $-p$, and use (1.3) and (1.6); we get after a little manipulation

$$B_{p-1}^{(-p)} - B_{p-1}^{(p)} \equiv \frac{2pB_{p-1}}{p-1} \pmod{p^3},$$

which is equivalent to (1.11).

3. **Proof of (1.13).** We shall require the formula [6, p. 146]

$$C_{m+1}^{(k)} = -k\sum_{s=0}^{m}(-1)^s\binom{m}{s}C_sC_{m-s}^{(k)}, \tag{3.1}$$

which is evidently analogous to (2.3). For $k=m+1=p$, (3.1) implies (since $C_{2r}=0$ for $r>0$)

$$\begin{aligned}\frac{1}{p}C_p^{(p)} &= -C_{p-1}^{(p)}+\sum_{s=1}^{p-4}\binom{p-1}{s}C_sC_{p-1-s}^{(p)}+(p-1)C_{p-2}C_1^{(p)}\\ &\equiv -C_{p-1}^{(p)}-p(p-1)C_{p-2}\\ &\quad -p^2(p-1)\sum_{1}^{p-4}\binom{p-2}{s}C_sC_{p-3-s} \pmod{p^3}\end{aligned} \tag{3.2}$$

by the second of (1.7). In the next place if we take $m+1=p-1$ in (3.1), we get

$$\begin{aligned}C_{p-1}^{(p)} &= -pC_{p-2}^{(p)}+p\sum_{1}^{p-4}\binom{p-2}{s}C_sC_{p-2-s}^{(p)}+pC_{p-2}C_0^{(p)}\\ &\equiv p^2\sum_{1}^{p-4}\binom{p-2}{s}C_sC_{p-3-s} \pmod{p^3},\end{aligned} \tag{3.3}$$

by the first of (1.7). Comparison of (3.2) and (3.3) yields

$$\begin{aligned}\frac{1}{p}C_p^{(p)} &\equiv -pC_{p-1}^{(p)}-p(p-1)C_{p-2}\\ &\equiv -p(p-1)C_{p-2} \pmod{p^3},\end{aligned}$$

which is equivalent to (1.13).

4. **Proof of (1.14) and (1.15).** We remark first that for $k>0$ we have

$$2^{k-m}C_m^{(-k)} = \sum_{s=0}^{k}\binom{k}{s}s^m, \tag{4.1}$$

by means of which it is easy to prove [5, p. 290]

$$2^{k-m}C_m^{(-k)} = \sum_{s=0}^{m}(-1)^s\binom{m}{s}2^{k-s}k^{m-s}C_s^{(-k)}. \tag{4.2}$$

In (4.1) and (4.2) we take

$$k = p = m;$$

then (4.2) becomes

$$
\begin{aligned}
2C_p^{(-p)} &= \sum_{s=1}^{p} (-1)^{p-s} \binom{p}{s} 2^s p^s C_{p-s}^{(-p)} \\
&\equiv 2p^2 C_{p-1}^{(-p)} - 2p^3 (p-1) C_{p-2}^{(-p)} \\
&\quad + \tfrac{4}{3} p^4 (p-1)(p-2) C_{p-3}^{(-p)} \\
&\equiv 2p^2 C_{p-1}^{(-p)} - 2p^5 (p-1)(p-2) C_{p-4} \\
&\quad - \tfrac{4}{3} p^5 (p-1)(p-2) C_{p-4} \\
&\equiv 2p^2 C_{p-1}^{(-p)} + \tfrac{4}{3} p^5 C_{p-4} \pmod{p^6},
\end{aligned}
\tag{4.3}
$$

where we have used (1.8). This proves (1.14).

In the next place if we take $k = -p$, $m+1 = p-1$ in (3.1) we get

$$
\begin{aligned}
C_{p-1}^{(-p)} &= p \sum_{s=0}^{p-2} (-1)^s \binom{p-2}{s} C_s C_{p-2-s}^{(-p)} \\
&= -pC_{p-2} + pC_{p-2}^{(-p)} - p \sum_{1}^{p-4} \binom{p-2}{s} C_s C_{p-2-s}^{(-p)} \\
&\equiv -pC_{p-2} + p^2 \sum_{1}^{p-4} \binom{p-2}{s} C_s C_{p-3-s} \pmod{p^3}.
\end{aligned}
\tag{4.4}
$$

Comparison of (4.4) with (3.3) gives

$$C_{p-1}^{(-p)} \equiv C_{p-1}^{(p)} - pC_{p-2} \pmod{p^3}, \tag{4.5}$$

which implies the first half of (1.15). The second half follows from (4.4).

5. Some of the above formulas can be simplified slightly. For example it is easy to show that

$$\sum_{1}^{p-4} \binom{p-2}{s} C_s C_{p-3-s} \equiv \frac{1}{2} \sum_{1}^{p-4} C_s C_{p-3-s} \pmod{p}. \tag{5.1}$$

It is not evident whether the right member of (5.1) can be reduced further. In this connection the formula

$$C_{m+1} = \sum_{1}^{m-1} \binom{m}{r} C_r C_{m-r} \tag{5.2}$$

may be mentioned. It follows from (5.2) that

$$C_p \equiv -\sum_{1}^{p-2} C_r C_{p-1-r} \pmod{p}.$$

In the second place we recall that [6, p. 28]

$$C_{m-1} = 2^m(1-2^m)\frac{B_m}{m}\cdot \tag{5.3}$$

By means of (5.3), (1.13) for example becomes

$$C_p^{(p)} \equiv 2^{p-1}(2^{p-1}-1)p^2 B_{p-1} \pmod{p^4},$$

while (1.14) becomes

$$C_p^{(-p)} \equiv p^2 C_{p-1}^{(-p)} - \frac{1}{24} p^5 B_{p-3} \pmod{p^6},$$

and so on.

References

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