SOME COUNTEREXAMPLES IN THE CLASSIFICATION OF OPEN RIEMANN SURFACES[1]

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Recently Ahlfors and the author [1] constructed a Riemann surface of hyperbolic type which possessed no nonconstant harmonic functions with a finite Dirichlet integral. In the first section we explore some of the consequences of this example and construct a Riemann surface on which the spaces HD and HBD have dimension n. In the next section a bounded Riemann surface is exhibited which has no HD functions on it which vanish on the relative boundary, while it has a nonconstant HD whose normal derivative vanishes on the relative boundary. In the last section we use a refinement of the method in [1] to construct a Riemann surface admitting a nonconstant bounded harmonic function, but no nonconstant harmonic functions with a finite Dirichlet integral, thus demonstrating that the classes O_{HB} and O_{HD} are distinct.

1. Consider a sequence $1/2 < r_1 < \cdots < r_n < \cdots < 1$ and the segments

$$\Delta_n^h: \quad \begin{cases} r_n \leqq r \leqq r_{n+1}, \\ \theta = 2\pi h \cdot 2^{-n}, \end{cases} \qquad 0 \leqq h < 2^n.$$

We divide each Δ_n^h into 2^n subsegments $\Delta_n^{h,k}$ of equal logarithmic length and form a Riemann surface W by identifying the left edge of $\Delta_n^{h,k}$ with the right edge of $\Delta_n^{h+k,k}$, where it is to be understood that $h+k$ is reduced to its remainder mod 2^n. It was shown in [1] that W has no nonconstant harmonic functions with a finite Dirichlet integral defined on it, provided

$$\sup 2^n \log \frac{1}{r_n} = \infty.$$

Let V be the surface formed by removing the circle $K: r < 1/2$ from W, and denote the circumference $r = 1/2$ by R. We use $HN = HN(V)$ to denote the space of harmonic functions on V which have a finite Dirichlet integral and whose normal derivative vanishes on R and use $HO = HO(V)$ to denote the space of those functions which have a finite Dirichlet integral and which vanish on R.

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By Theorem 10 of [2] the space $HBD(W)$ of those harmonic functions on W which have a finite Dirichlet integral and are bounded is isomorphic to

$$HO(V) \dotplus HO(K)$$

and to

$$HN(V) \dotplus HN(K).$$

Since W has a Green's function, we include the constants in HBD by convention. Hence HBD consists of the constants alone and is thus one-dimensional. The compactness of K implies that $HO(K)$ and $HN(K)$ contain only the function zero. Remembering that HBD is dense in HD in the sense of the Dirichlet metric [2], we also have HD one-dimensional and hence:

The spaces $HO(V)$ and $HN(V)$ have dimension one. Consequently $HO(V)$ consists of multiples of $\log (r/2)$ *while $HN(V)$ consists of constants.*

If we reflect V in the circle R, we obtain a surface W_2 on which the space HD has dimension 2. Thus if u is harmonic and has a finite Dirichlet integral we must have

$$u = C_1 + C_2 \log r.$$

Similarly, if we start from the complex sphere from which $n+1$ circular disks have been removed and attach a replica of V in place of each disk, we obtain a surface W_n with the property that $HBD(W_n)$ is n-dimensional. Thus:

For each integer n there exists a Riemann surface W_n on which the spaces $HD(W_n)$ and $HBD(W_n)$ are n-dimensional.

2. Consider a sequence $0<\eta_1< \cdots <\eta_n< \cdots <1$, set $\eta_{-n} = -\eta_n$, and form the segments

$$\Delta_n^h: \begin{cases} \eta_n \leqq y \leqq \eta_{n+1}, \\ x = (h+1)2^{-|n|}, \qquad 0 \leqq h \leqq 2^{|n|} - 1. \end{cases}$$

We divide each Δ_n^h into 2^n subsegments $\Delta_n^{h,k}$ of equal lengths and construct a bounded Riemann surface V' from the interior of the rectangle $|y| \leqq 1$, $0 \leqq x \leqq 1$, by identifying the left edge of $\Delta_n^{h,k}$ with the right edge of $\Delta_n^{h+k,k}$, where here it is understood that $h+k$ is reduced to its remainder mod $2^{|n|}-1$. The relative boundary R of V' consists of the segments $x=0$ and $x=1$.

Let u be a harmonic function defined on V' which has a finite Dirichlet integral and vanishes on R. Then, given $\delta>0$, there is a set

E_δ of measure greater than $1-\delta$ such that

$$\int_{-1}^{1}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dy < M_\delta$$

for $x\in E_\delta$. Now

$$|u(x, y_2) - u(x, y_1)|^2 = \left(\int_{y_1}^{y_2}\frac{\partial u}{\partial y}dy\right)^2 \leqq (y_2 - y_1)M_\delta$$

by the Schwarz inequality, and hence there is a function $l(x)$ such that

$$\lim_{y\to 1} u(x, y) = l(x)$$

uniformly for $x\in E_\delta$. Thus for $\epsilon>0$, we may choose N so that $l(x)^2 \leqq u(x, y)^2+\epsilon$ for all $x\in E_\delta$ and all $y\geqq \eta_N$.

Since the right edge of the interval $\Delta_n^{h,h}$ is identified with the left edge of $\Delta_n^{0,h}$, we have

$$u(x, y) = \int_0^{2^{-n}}\frac{\partial u}{\partial x}dx + \int_{(h+1)2^{-n}}^{x}\frac{\partial u}{\partial x}dx$$

for $(h+1)2^{-n}\leqq x\leqq (h+2)2^{-n}$ and for y in the projection of $\Delta_n^{0,h}$. By the Schwarz inequality

$$u(x, y)^2 \leqq 2^{-n}\int_0^{2^{-n}}\left(\frac{\partial u}{\partial x}\right)^2 dx + 2^{-n}\int_{(n+1)2^{-n}}^{x}\left(\frac{\partial u}{\partial x}\right)^2 dx,$$

whence

$$u^2 \leqq 2^{-n+1}\int_0^1\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dx.$$

For $n\geqq N$ and x in I_n, the intersection of E_δ with the interval

$$(h + 1)2^{-n} \leqq x \leqq (h + 2)2^{-n},$$

we have also

$$l(x)^2 \leqq 2^{-n+1}\int_0^1\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dx + \epsilon.$$

Integrating with respect to y in the projection P^h of $\Delta_n^{0,h}$ gives

$$2^{-n}(\eta_{n+1} - \eta_n)l(x)^2 \leqq 2^{-n+1}\int_{P^h}\int_0^1\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dxdy$$
$$+ 2^{-n}(\eta_{n+1} - \eta_n).$$

Integrating with respect to h in I_n gives

$$2^{-n}(\eta_{n+1} - \eta_n) \int_{I_n} l(x)^2 dx \leqq 2^{-2n+1} \int_{P^h} \int_0^1 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dxdy + \epsilon 2^{-2n}(\eta_{n+1} - \eta_n).$$

Thus (if we put $l = 0$ outside E_δ)

$$\begin{aligned}(\eta_{n+1} - \eta_n) \int_{2^{-N}}^{1-2^{-N}} l^2 dx &\leqq \sum_{h=1}^{2^n-2} (\eta_{n+1} - \eta_n) \int_{I_h} l^2 dx \\ &\leqq 2^{-n+1} \int_{\eta_n}^{\eta_{n+1}} \int_0^1 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dxdy \\ &\quad + \epsilon(\eta_{n+1} - \eta_n).\end{aligned}$$

Therefore

$$(\eta_{n+1} - \eta_n) \int_{2^{-N}}^{1-2^{-N}} l^2 dx \leqq 2^{-n+1} D(u) + \epsilon(\eta_{n+1} - \eta_n).$$

Summing for $n \geqq N$ gives

$$(1 - \eta_N) \int_{2^{-N}}^{1-2^{-N}} l^2 dx \leqq 2^{-N+2} D(u) + \epsilon(1 - \eta_N).$$

If we let η_n converge to one so slowly that

$$\sup 2^n(1 - \eta_n) = \infty,$$

then

$$\int_0^1 l^2 dx < \epsilon,$$

and since the left-hand side is independent of n,

$$\int_0^1 l^2 dx = 0,$$

hence $l = 0$ almost everywhere in E_δ. Since δ is arbitrary we must have

$$\lim_{y \to 1} u(x, y) = 0$$

for almost all x in $[0, 1]$. Similarly

$$\lim_{y \to -1} u(x, y) = 0$$

for almost all x in $[0, 1]$.

For almost all x we then have

$$u(x, y) = \int_{-1}^{y} \frac{\partial u}{\partial y} dy$$

and by the Schwarz inequality

$$u(x, y)^2 \leqq 2\int_{-1}^{1}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dy = \mu(x).$$

But μ is summable since

$$\int_0^1 \mu(x)dx = 2D(u).$$

Hence

$$m(y) = \int_0^1 u^2 dx$$

is continuous and $m(1) = m(-1) = 0$. Moreover

$$\begin{aligned} m'(y_2) - m'(y_1) &= 2\int_0^1 u \frac{\partial u}{\partial y} dx \Big|_{y=y_2} - 2\int_0^1 u \frac{\partial u}{\partial y} dx \Big|_{y=y_1} \\ &= 2\int_{y_1}^{y_2}\int_0^1 \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy, \end{aligned}$$

since u is a harmonic function which vanishes for $x = 0$ and $x = 1$. Thus $m'(y)$ is increasing whence $m(y)$ is convex. But $m(y)$ must vanish identically in the interval $-1 \leqq y \leqq 1$, since it is non-negative, convex, and vanishes at the end points. Hence $u \equiv 0$.

The function $u = y$ is single-valued on V' and has a finite Dirichlet integral, while $\partial u/\partial n = 0$ on R. Thus we have the following result.

On the bounded Riemann surface V' the class HO is empty, while there is a nonconstant harmonic function with a finite Dirichlet integral whose normal derivative vanishes on the relative boundary.

3. We form a Riemann surface W' in the strip $|y| \leqq 1$ by putting in replicas of V' in each rectangle $n \leqq x \leqq n+1$. Let u be a bounded harmonic function with a finite Dirichlet integral defined on W'. Then a slight modification in the argument of the preceding section shows that

$$\lim_{y \to 1} u(x, y) = c$$

for almost all x, and

$$\lim_{y \to -1} u(x, y) = c'$$

for almost all x.

Without loss of generality we may take $c' = 0$. Then

$$u(x, y) = \int_{-1}^{y} \frac{\partial u}{\partial y} dy$$

and by the Schwarz inequality

$$u(x, y)^2 \leqq 2 \int_{-1}^{y} \left(\frac{\partial u}{\partial y}\right)^2 dy \leqq 2 \int_{-1}^{1} \left(\frac{\partial u}{\partial y}\right)^2 dy = \mu(x).$$

Since

$$\int_{-\infty}^{\infty} \mu(x) \leqq D(u) < \infty,$$

μ is summable and so

$$m(y) = \int_{-\infty}^{\infty} u(x, y)^2 dx$$

exists and is continuous for $-1 \leqq y \leqq 1$. Also we have $c^2 \leqq \mu(x)$, whence c must be zero in order for μ to be summable on $(-\infty, \infty)$. Hence, $m(y_1) = m(y_2) = 0$. By the Schwarz inequality we have

$$\left(\int\int \left| u \frac{\partial u}{\partial y} \right| dxdy\right)^2 \leqq \int\int u^2 dxdy \int\int \left(\frac{\partial u}{\partial y}\right)^2 dxdy$$
$$\leqq D(u)^2 < \infty.$$

Thus by the Fubini theorem

$$m_1(y) = \frac{1}{2} \int_{-\infty}^{\infty} u \frac{\partial u}{\partial y} dx$$

exists for almost all y and

$$m(y_2) - m(y_1) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{y_1}^{y_2} u \frac{\partial u}{\partial y} dydx$$
$$= \frac{1}{2} \int_{y_1}^{y_2} \int_{-\infty}^{\infty} u \frac{\partial u}{\partial y} dxdy$$
$$= \int_{y_1}^{y_2} m_1(y) dy,$$

whence m is absolutely continuous and

$$m' = m_1 \quad \text{a.e.}$$

Let y_1 and y_2 be two values of y for which m_1 exists, and take N so large that $y_1 < \eta_N$, $y_2 < \eta_N$. Since

$$\iint \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dxdy < \infty,$$

we can find an arbitrarily large x_0 such that the fractional part of x_0 lies between zero and 2^{-N} and such that

$$\int \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dy < \frac{\epsilon^2}{2}$$

on the two segments $x = \pm x_0$. On these segments

$$\left(\int \left|\frac{\partial u}{\partial x}\right| dy\right)^2 \leqq 2\int \left(\frac{\partial u}{\partial x}\right)^2 dy < \epsilon^2$$

and so

$$\int \left|\frac{\partial u}{\partial x}\right| dy < \epsilon.$$

We also take x_0 so large that

$$\iint_{|x|>x_0} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy < \epsilon$$

and

$$\int_{|x|>x_0} \left| u \frac{\partial u}{\partial y}\right| dx < \qquad \text{for } y = y_1, y_2.$$

Since the fractional part of x_0 lies between zero and 2^{-N}, the region Ω on W' for which $|x| < x_0$ and $y_1 \leqq y \leqq y_2$ has as its boundary the segments $|x| < x_0$, $y = y_1$, y_2 and the segments $y_1 < y < y_2$, $|x| = x_0$. Thus by Green's theorem

$$D_\Omega(u) = \int u \frac{\partial u}{\partial n} = \int_{x=x_0} u \frac{\partial u}{\partial x} dy - \int_{x=-x_0} u \frac{\partial u}{\partial x} dy$$
$$+ \int_{y=y_2} u \frac{\partial u}{\partial x} dy - \int_{y=y_1} u \frac{\partial u}{\partial x} dy$$

where the ranges of integration are $y_1 \leqq y \leqq y_2$, $-x_0 \leqq x \leqq x_0$.

Thus

$$\left| D_\Omega(u) - m_1(y_2) + m_1(y_2) \right| < 2M\epsilon + 2\epsilon$$

where M is a bound for $|u|$. Consequently

$$\left| \int_{y_1}^{y_2} \int_{-\infty}^{\infty} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dxdy - m_1(y_2) + m_1(y_1) \right| \quad < (2M + 3)\epsilon.$$

The left-hand side must be zero since it is independent of ϵ, and so

$$m_1(y_2) - m_1(y_1) = \int_{y_1}^{y_2} \int_{-\infty}^{\infty} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dxdy \geqq 0.$$

From this and the fact that

$$m(y) = \int^{y} m_1(y)dy$$

we conclude that $m(y)$ is convex. As a result $m(y) \equiv 0$ in the interval $-1 \leqq y \leqq 1$, since it is a non-negative convex function vanishing at the ends of the interval. This implies $u \equiv 0$, and because of the identity $O_{HD} = O_{HB}$ we have the following proposition:

The Riemann surface W' has no nonconstant harmonic functions on it with a finite Dirichlet integral, while the function $u = y$ is a harmonic function which is defined, single-valued, and bounded on W'.

Bibliography

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