CONVERSE OF A WELL KNOWN THEOREM ON INTEGRAL MEANS¹

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0. **Introduction.** Let $f(x)$ be a real-valued function defined and integrable, in the sense of Lebesgue, in an open interval I of real numbers. If, for $h \neq 0$, the points $x, x+h$ lie in I , the integral of f from x to $x+h$, divided by h , is called the integral mean of length h of f at x , and will be denoted by $M_h f(x)$.

The integral mean of a function has been a useful tool in various investigations due to the fact that it is "smoother" than the function itself; for example, if f is integrable in I , its mean will be continuous,² while if f has a continuous derivative of order $n \geq 0$, the mean will have a continuous derivative of order at least $n+1$.

The theorem which will be proved is the converse, in a sense, of the above result. We shall show, under a certain restriction (that some restriction is required is evident from simple examples), that the smoothing power of the operation of taking the integral mean is limited; i.e., if f is integrable but not continuous, $M_h f$ will be continuous but not continuously differentiable, while if f has a continuous derivative of order at most n , $M_h f$ will have a continuous derivative of order at most $n+1$.

1. **Differences, difference quotients, and means.** Frequent use will be made of the difference, difference quotient, and mean operators, together with certain commutativity relations satisfied by them.

Let the difference of length h of a function f at x be given by

$$(1) \quad \Delta_h f(x) = \Delta f(x, x+h) = f(x+h) - f(x)$$

and the difference quotient operator of length h at x by

$$(2) \quad D_h f(x) = Df(x, x+h) = \frac{f(x+h) - f(x)}{h}.$$

The mean of length h of an integrable function f at x is defined by the formula

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² It is to be understood in all that follows that a condition on $M_h f$ applies in the subinterval of I in which it is defined.

$$(3) \quad M_h f(x) = \frac{1}{h} \int_x^{x+h} f(\theta) d\theta.$$

Clearly, each operator of length h commutes with all operators of length k in the subinterval of definition of the function f in which the iterated operators are defined. In addition, it is easily verified that

$$(4) \quad D_h M_k = D_k M_h.$$

If the function f is continuous, the relations

$$(5) \quad \lim_h \Delta_h f = 0,$$

$$(6) \quad \lim_h M_h f = f$$

hold in I . By relations (4) and (6)

$$DM_h f = \lim_k D_k M_h f = \lim_k D_h M_k f = D_h \lim_k M_k f = D_h f.$$

Therefore, if f is continuous,

$$(7) \quad DM_h f = D_h f.$$

Suppose Df exists and is continuous in I . Then from the fundamental theorem of integral calculus, it follows, for $x, x+h$ in I , that

$$\int_x^{x+h} Df(\theta) d\theta = \Delta_h f(x) = h D_h f(x).$$

Therefore, using relations (3) and (7),

$$(8) \quad M_h Df = DM_h f$$

if Df is continuous.

2. On the first derivative of the mean. Relation (7) of §1 shows that if f is continuous in I , the derivative of the mean exists and is continuous. Relation (7) still holds, however, if the assumption of continuity of f is replaced by the weaker hypothesis that $\lim_h M_h f = f$. A function f satisfying in the interval I this weaker hypothesis will be called mean continuous (MC) in I .

It is for MC functions that the theorem stated in the introduction will be proved. The assumption of mean continuity is a natural one to make in view of the fact that relation (7), which is basic to the proofs that follow, holds in its presence. In addition, the theorem is false if the function f is assumed only to be integrable. To see this, one need merely consider any discontinuous function which is con-

stant almost everywhere.

If f is MC in I and for all $h \neq 0$ the derivative $DM_h f$ is continuous, then by relation (7), $D_h f$, and thus $\Delta_h f$, is continuous. Consider the identity

$$(9) \quad \Delta_k \Delta_h f(x) + \Delta_k f(x) = \Delta_k f(x + h).$$

Since, by (5), $\lim_k \Delta_k \Delta_h f(x) = 0$, it is clear that if $\lim_k \Delta_k f(x) = 0$ for a single point x in I , then $\lim_k \Delta_k f(y) = 0$ for every point $y = x + h$ in I . Thus f is either continuous in I or discontinuous at every point in I . Since $\lim_h M_h f = f$, Baire's theorem on the limit of continuous functions guarantees that f is continuous on a set of second category. Therefore, an MC function f whose mean has, for all $h \neq 0$, a continuous derivative $DM_h f$ is continuous.

3. On the second derivative of the mean. Suppose f is MC in I and, for all $h \neq 0$, the second derivative of the mean $D^2 M_h f$ exists and is finite. Consider the identity

$$(10) \quad D_k \Delta_h f(x) + D_k f(x) = D_k f(x + h).$$

Since, by relation (7),

$$\lim_k D_k \Delta_h f(x) = h \lim_k D_k D_h f = h \lim_k D_k D M_h f = h D^2 M_h f$$

exists for all h , whenever $x, x+h$ are in I , it follows from (10) that if f has one of its unilateral derivates, say the upper right derivate $*D^+ f = \limsup_{h \rightarrow 0^+} D_h f$, finite at a point x , then $*D^+ f(y)$ is finite for every $y = x + h$ in I . If one of the derivates of f is infinite at a single point in I , it must be infinite of the same sign everywhere in I . Moreover, if Df exists at a single point in I , f is derivable in I and either Df is finite or infinite of the same sign everywhere in I .

A finite function f possessing this behavior will be called a same derivative behavior, or SDB, function in I . To summarize, a finite function f is an SDB function in I if

(i) each unilateral derivate of f is either finite in I or infinite of the same sign in I ;

(ii) whenever Df exists at a single point, f is differentiable in I , and Df is either finite throughout I or infinite of the same sign throughout I .

An SDB function for which all the unilateral derivates are infinite will be called an extreme function. For extreme functions f ,

$$*D^+ f = *D^- f = +\infty, \quad *D^+ f = *D^- f = -\infty.$$

Indeed, if one of the upper derivates, say $*D^+ f$, satisfies $*D^+ f = -\infty$,

then $*D^+f \leq *D^-f$ implies that f has an infinite right derivative everywhere in I . It is known, however, that a finite function f can have an infinite unilateral derivative on at most a set of measure zero [3].³ Thus it is also clear that if an SDB function has a derivative in I , that derivative is finite in I .

We shall prove in §5 that an SDB function is either differentiable or extreme. Further, we shall show that a continuous SDB function cannot be extreme, hence is differentiable.

Making use of these results on SDB functions, it is easy to prove that if f is MC in I and, for all $h \neq 0$, the second derivative of the mean $D^2M_h f$ exists and is continuous, then Df exists and is continuous in I . Since $D^2M_h f$ exists, relation (10) shows that f is an SDB function. Moreover, since $DM_h f$ must be continuous, the result of §2 shows that f is continuous. Thus f , being a continuous SDB function, is differentiable. To see that Df is continuous, note that Df is the limit of the continuous functions $D_h f$, while

$$hD^2M_h f = hDD_h f = hD_h Df = \Delta_h Df$$

is continuous, and therefore the situation is analogous to that occurring in §2.

4. On the n th derivative of the mean. Using the results of §§2 and 3, the main theorem is proved by induction.

MAIN THEOREM. *Let the integrable function f be MC in the open interval I . Then if, for all $h \neq 0$, $M_h f$ has a continuous derivative of order $n+1$, $n \geq 0$, f has a continuous derivative of order n in I .*

PROOF. The truth of the theorem for $n=0, 1$ has been established in §§2, 3. Assume the theorem is true for $n-1 \geq 0$: to prove it for n . If for all $h \neq 0$, $D^{n+1}M_h f$ is continuous, it follows that $D^2M_h f$ is continuous, and therefore Df is continuous in I . By virtue of relation (8),

$$D^{n+1}M_h f = D^n(DM_h f) = D^n M_h Df.$$

Thus Df is MC while, for all $h \neq 0$, its mean $M_h Df$ has a continuous derivative of order n . By assumption, $D^{n-1}(Df) = D^n f$ exists and is continuous in I , so by induction the theorem is proved.

5. Theorems on SDB functions.

THEOREM 1. *If f is an SDB function in I , f is extreme or f is differentiable in I .*

In order to prove Theorem 1 we make use of the following

³ Numbers in brackets refer to the bibliography.

LEMMA. If a finite function f defined on an open interval I has its upper right derivate $*D^+f < \infty$ on I , then its upper derivate $*Df = \limsup_{h \rightarrow 0} D_h f$ is uniformly bounded from above on some open subinterval of I .

PROOF. For each point x in I , let $I_n(x)$ be that open subinterval of I consisting of all points $y > x$ in I which are closer than $1/n$ to x . Let F_n be the set of points x of I such that

$$Df(x, y) = \frac{f(y) - f(x)}{y - x} \leq n$$

for every y in $I_n(x)$. We shall show that F_n is closed from the right.

Suppose, to the contrary, that there exist points x_m , x in I with $x_m \rightarrow x^+$, x_m in F_n , but x not in F_n . Therefore, $Df(x, y) > n$ for some point y in $I_n(x)$, and so has the value $n + \epsilon$, $\epsilon > 0$. Moreover since y lies in $I_n(x)$ and $x_m \rightarrow x^+$, y also lies in $I_n(x_m)$ for almost all m . Hence for all such m , we have $Df(x_m, y) \leq n$ since x_m belongs to F_n . Now

$$f(x_m) - f(x) = [f(y) - f(x)] - [f(y) - f(x_m)]$$

or

$$\begin{aligned} \Delta f(x, x_m) &= \Delta f(x, y) - \Delta f(x_m, y) \\ &= (y - x)Df(x, y) - (y - x_m)Df(x_m, y). \end{aligned}$$

Therefore, for almost all m ,

$$\Delta f(x, x_m) \geq (y - x)(n + \epsilon) - (y - x)n = (y - x)\epsilon,$$

whereupon

$$*D^+f(x) \geq \lim_{m} Df(x, x_m) = +\infty,$$

contrary to $*D^+f(x) < +\infty$. This proves F_n closed from the right.

Every point x in I is in some set F_n . If this were not the case, some x in I would belong to no set F_n ; thus for each n , there would be a point y_n in $I_n(x)$ with $Df(x, y_n) > n$, whereupon, since $y_n \rightarrow x^+$,

$$*D^+f(x) \geq \lim_{n} Df(x, y_n) = +\infty$$

contradicting $*D^+f(x) < +\infty$. Thus I is the union of the sets F_n . According to Baire's theorem, at least one of the sets F_n must be somewhere dense—dense, say, in the open subinterval J of I . But this set F_n , being closed from the right, must contain J . Consequently,

for any two distinct points x, y of J closer than $1/n$ we have $Df(x, y) \leq n$, so

$$*Df(x) = \limsup_{y \rightarrow x} Df(x, y) \leq n$$

for every x in J . This completes the proof of the lemma.

PROOF OF THEOREM 1. If f is not extreme, one of its derivatives, say $*D^+f$, is finite in I . Under this condition, it has been proved (A. Denjoy [1] for continuous functions; S. Saks [2] for arbitrary finite functions) that $*D^+f = *D^-f$ almost everywhere in I . The preceding lemma shows that $*D^-f$ is also finite in I , whence $*D^-f = *D^+f$ almost everywhere. Thus Df exists and is finite almost everywhere in I , and f , assumed to be a nonextreme SDB function, must therefore be differentiable.

By virtue of Theorem 1, it is necessary only to prove that a continuous function cannot be extreme in order to establish

THEOREM 2. *If f is a continuous SDB function in I , f is differentiable in I .*

PROOF. Let a, b be any two points in I . The continuous function

$$g(x) = f(x) - (x - a)Df(a, b)$$

has the same differentiability properties as f . Since $g(a) = g(b)$, g either is constant in the interval $[a, b]$ or assumes an extreme value at some point y in the interior. Thus one, at least, of the upper derivatives of g at y is nonpositive, and g cannot be extreme.

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