AN ADDITION THEOREM FOR SETS OF ELEMENTS OF ABELIAN GROUPS

H. B. MANN

Let G be a finite Abelian group, A, B sets of elements in G. By AB we denote the set of all elements of the form ab ($a\in A$, $b\in B$), by $\bar{A}$ the complement of A in G, and by (A) the number of elements in A. We shall prove the following theorem.

THEOREM. *If for the set A and every subgroup H of G*

$$(1) \qquad (AH) \geqq (A) + (H) - 1 \quad \textit{or} \quad AH = G,$$

then for every set B

$$(2) \qquad (AB) \geqq (A) + (B) - 1 \quad \textit{or} \quad AB = G.$$

For the proof we need the following two theorems proved in [1].

(i) If A, B are sets of elements of a finite group G, then

$$(3) \qquad (G) \geqq (A) + (B) \quad \text{or} \quad G = AB.$$

(ii) If G is a finite Abelian group $c\in G$, $c\notin AB$, then there exists a set $B^*\subset G$ such that

(1) $B^*\supseteq B$,

(2) $\overline{AB^*}=cH$, where H is a sub-group of G,

(3) $(AB^*)-(AB)=(B^*)-(B)$.

PROOF OF THE THEOREM. By virtue of (ii) it is sufficient to prove the theorem for the case that $\overline{AB}=cH$ where H is a subgroup of G. Consider then the factor group G/H and let A', B' be the set of cosets of H contained in AH, BH respectively. Then by (i)

$$(4) \qquad (G/H) \geqq (A') + (B').$$

But $(G/H)(H)=(G)$, $(B')(H)\geqq(B)$, and by hypothesis $(A')(H) =(AH)\geqq(A)+(H)-1$. Multiplying (4) by (H) we therefore get

$$(G) \geqq (A) + (B) + (H) - 1$$

and

$$(AB) = (G) - (H) \geqq (A) + (B) - 1.$$

REFERENCE

1. H. B. Mann, *On products of sets of group elements*, Canadian Journal of Mathematics vol. 4 (1952) pp. 64–66.

THE OHIO STATE UNIVERSITY

Received by the editors August 11, 1952.