ON UNIFORM DISTRIBUTION OF ALGEBRAIC NUMBERS

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Let K be an algebraic number field of degree $n+1$ over the rational field $\mathbb{R}$. The conjugates of K are denoted by $K^{(0)}, \dots, K^{(n)}$, it being supposed that $K^{(0)}, \dots, K^{(r_1)}$ are real, $K^{(r_1+1)}, \dots, K^{(n)}$ are complex, and moreover that for $k=r_1+1, \dots, r_1+r_2$ the fields $K^{(k)}$ and $K^{(k+r_2)}$ are obtainable from one another by interchanging $i(=(-1)^{1/2})$ and $-i$ (here $r_1+2r_2=n$). I assume throughout that $r_1 \geq 0$ and that $n \geq 1$. Numbers in K are represented by small Greek letters, the insertion of superscripts denoting the passage to the corresponding conjugates. Let $\omega_0, \dots, \omega_n$ be a set of numbers in K which are linearly independent over $\mathbb{R}$; let $\omega_0=1$; and finally let $f(x_1, \dots, x_n)$ be a (complex-valued) function of n real variables which is Riemann integrable over the unit cube $E_n: 0 < x_k < 1$ ($k=1, \dots, n$). Then, according to theorems of Weyl on uniform distribution,

$$(1) \quad \int_{E_n} f(x_1, \dots, x_n) dx_1 \dots dx_n = \frac{1}{M} \sum_{m=0}^{M-1} f(\{m\omega_1^{(0)}\}, \dots, \{m\omega_n^{(0)}\}) + o(1)$$

as $M \rightarrow \infty$, the braces denoting the fractional part. The object of this paper is to show that the error term in (1) may be replaced by $O(1/M)$ provided f satisfies certain additional conditions.

More precisely, suppose that in E_n

$$(2) \quad f(x_1, \dots, x_n) = \sum_{q_1, \dots, q_n = -\infty}^{\infty} a(q_1, \dots, q_n) e^{2i\pi(q_1 x_1 + \dots + q_n x_n)},$$

the series on the right being absolutely convergent.

THEOREM. *If there exist positive constants C and c such that*

$$(3) \quad |a(q_1, \dots, q_n)| < C(|q_1| + \dots + |q_n|)^{-n-c},$$

then the expression

$$(4) \quad S_M = M \int_{E_n} f(x_1, \dots, x_n) dx_1 \dots dx_n - \sum_{m=0}^{M-1} f(m\omega_1^{(0)}, \dots, m\omega_n^{(0)})$$

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is bounded in absolute value with a bound which depends only on $C, c, \omega_1, \dots, \omega_n$ (and not on M).

The theorem probably remains true when (3) is replaced by the weaker condition

$$|a(q_1, \dots, q_n)| < C(1 + |q_1|)^{-1-c} \dots (1 + |q_n|)^{-1-c},$$

but I have been unable to prove this.

For the proof of the theorem, it is clear from (2) and (4) that

$$S_M = \sum'_{q_1, \dots, q_n} a(q_1, \dots, q_n) \frac{\sin \pi M(q_1 \omega_1^{(0)} + \dots + q_n \omega_n^{(0)})}{\sin \pi (q_1 \omega_1^{(0)} + \dots + q_n \omega_n^{(0)})} \cdot e^{i\pi(M-1)(q_1 \omega_1^{(0)} + \dots + q_n \omega_n^{(0)})}$$

where the ' indicates that $q_1 = \dots = q_n = 0$ is to be omitted in summation. Thus, from (3),

$$|S_M| < \frac{C}{2} \sum' (|q_1| + \dots + |q_n|)^{-n-c} \cdot \|q_1 \omega_1^{(0)} + \dots + q_n \omega_n^{(0)}\|^{-1}$$

where $\|x\|$ stands for $\min_{k=0, \pm 1, \pm 2, \dots} |x - k|$.

Let $\mathfrak{A}$ denote the set of numbers α in K of the form

$$\alpha = q_0 + q_1 \omega_1 + \dots + q_n \omega_n$$

with rational integers $q_0, \dots, q_n$. An element α of $\mathfrak{A}$ is uniquely determined by the numbers $q_1, \dots, q_n$ and the condition

$$|\alpha^{(0)}| < 1/2.$$

When this condition is satisfied, one sees that

$$|q_0| < \max_{k=1, \dots, n} |\omega_k^{(0)}| \cdot (|q_1| + \dots + |q_n|) + 1/2,$$

i.e.,

$$|q_0| < C_1(|q_1| + \dots + |q_n|),$$

whence

$$|\alpha^{(k)}| < C_2(|q_1| + \dots + |q_n|) \quad (k = 1, \dots, n)$$

with

$$C_2 = 1/2 + \max_{k=1, \dots, n} |\omega_k^{(0)}| + \max_{j, k=1, \dots, n} |\omega_k^{(j)}|.$$

Thus,

$$|S_M| < C_3 \sum_{\alpha \text{ in } \mathfrak{A}, 0 < |\alpha^{(0)}| < 1/2} |N(\alpha)|^{-1} \left(\max_{k=1, \dots, n} |\alpha^{(k)}| \right)^{-c}.$$

The ideal $(1, \omega_1, \dots, \omega_n)$ may be written $\mathfrak{a}/m$ where $\mathfrak{a}$ is an integral ideal and m is a positive rational integer. The elements of $\mathfrak{A}$ are, of course, all divisible by $\mathfrak{a}/m$; hence

$$(5) \quad |S_M| < C_4 \sum_{\mathfrak{a} | \alpha, 0 < |\alpha^{(0)}| < m/2} |N(\alpha)|^{-1} \left(\max_{k=1, \dots, n} |\alpha^{(k)}| \right)^{-c}.$$

It is, therefore, sufficient to prove that the sum on the right side of (5), with $\mathfrak{a} = (1)$, is convergent.

Let P be a complete set of nonassociated integers of K , so that every integer of K is uniquely expressible as the product of a unit by an element of P . Then the sum in (5) with $\mathfrak{a} = (1)$ may be written

$$S = \sum_{\alpha \text{ in } P} |N(\alpha)|^{-1} \sum_{|N(\epsilon)|=1, 0 < |\epsilon^{(0)} \alpha^{(0)}| < m/2} \left(\max_{k=1, \dots, n} |\epsilon^{(k)} \alpha^{(k)}| \right)^{-c},$$

where ϵ , in the inner sum, runs over all units satisfying the indicated condition. If the elements α of P are chosen so that

$$C_5 |N(\alpha)|^{1/n} > \alpha^{(k)} > |N(\alpha)|^{1/n} \quad (k = 1, \dots, n)$$

(which according to the Dirichlet theory of units is always possible for a constant C_5 which depends only on the regulator of K), then

$$(6) \quad \begin{aligned} S &\leq \sum_{\alpha \text{ in } P} |N(\alpha)|^{-1-c/n} \sum_{|N(\epsilon)|=1, 0 < |\epsilon^{(0)} \alpha^{(0)}| < m/2} \left(\max_{k=1, \dots, n} |\epsilon^{(k)}| \right)^{-c} \\ &\leq \zeta_K \left(1 + \frac{c}{n} \right) \sum_{|N(\epsilon)|=1, |\epsilon^{(0)}| < C_6} \left(\max_{k=1, \dots, n} |\epsilon^{(k)}| \right)^{-c} \end{aligned}$$

where ζ_K is the Dirichlet zeta function for the field K and $C_6 = mC_5^n/2$ is a constant which depends only on the numbers $\omega_1, \dots, \omega_n$ and on the regulator of K .

Appealing once more to the Dirichlet theory of units, one sees from the $r(=r_1+r_2)$ -dimensional lattice formed by the logarithms of the units that there is a constant $C_7 > 0$ such that at most one unit ϵ (aside from multiplicative factors of roots of unity) satisfies

$$(7) \quad t_k \leq \log |\epsilon^{(k)}| < t_k + C_7 \quad (k = 1, \dots, n),$$

the real numbers $t_1, \dots, t_n$ being arbitrarily prescribed in advance with the convention that $t_{k+r_2} = t_k$ for $k = r_1 + 1, \dots, r_1 + r_2$. Note that

$$(8) \quad -nC_7 - (t_1 + \cdots + t_n) < \log |\epsilon^{(0)}| \leq - (t_1 + \cdots + t_n)$$

is a consequence of (7).

Since $r_1 \geq 0$, the only roots of unity in K are ± 1 ; therefore, choosing

$$t_k = m_k C_7 \quad (k = 1, \dots, r)$$

where $m_1, \dots, m_r$ are rational integers, one deduces from (6), (7), (8) that

$$S < 2\zeta_k \left(1 + \frac{c}{n}\right) \sum e^{-cC_7 \max(m_1, \dots, m_r)}$$

where the sum runs over all sets of rational integers $m_1, \dots, m_r$ such that $-(m_1 + \cdots + m_{r_1}) - 2(m_{r_1+1} + \cdots + m_{r_1+r_2}) < \log C_8/C_7 = -C_8$, say. Thus

$$S < 2\zeta_k \left(1 + \frac{c}{n}\right) r_1! r_2! \sum_{\substack{m_1 \leq m_2 \leq \dots \leq m_{r_1} \\ m_{r_1+1} \leq m_{r_1+2} \leq \dots \leq m_r \\ m_1 + \dots + m_{r_1} + 2(m_{r_1+1} + \dots + m_r) > C_8}} e^{-cC_7 \max(m_1, m_{r_1+1})}.$$

The latter sum may be divided into two sections in the first of which $m_1 \geq m_{r_1+1}$ and in the second of which $m_{r_1+1} > m_1$.

In the first section the inequalities under the summation sign imply $m_{r_1} > C_8 - (n-1)m_1$ and $m_r > (C_8 - (n-2)m_1)/2$ so that the number of summands corresponding to a particular m_1 is at most $(nm_1 - C_8)^{r-1} \cdot 2^{-r_2}$ or 0 according as $m_1 > C_8/n$ or $m_1 \leq C_8/n$. In the second section of the sum, the inequalities imply $m_{r_1} > C_8 - (n-1)m_{r_1+1}$ and $m_r > (C_8 - (n-2)m_{r_1+1})/2$, so that the number of summands corresponding to a particular m_{r_1+1} is at most $(nm_{r_1+1} - C_8)^{r-1} \cdot 2^{-r_2+1}$ or 0 according as $m_{r_1+1} > C_8/n$ or $m_{r_1+1} \leq C_8/n$. Thus,

$$(9) \quad S < 3 \cdot 2^{-r_2+1} r_1! r_2! \zeta_K \left(1 + \frac{c}{n}\right) \sum_{m=1+[C_8/n]}^{\infty} (nm - C_8)^{r-1} e^{-mcC_7}$$

which is clearly finite.

It is interesting to observe, from (9), that the bound for S is $O(c^{r-1})$.

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