FROM GENERALIZED CAUCHY-RIEMANN EQUATIONS TO LINEAR ALGEBRAS

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In a previous paper [1] the author gave a definition of analytic function in linear associative algebras with an identity. With each such algebra there was associated a set of partial differential equations called generalized Cauchy-Riemann differential equations which serve as a criterion to determine whether a given function is analytic in that algebra. A simplification for the commutative case has been given by Wagner [2].

The purpose of this paper is to give sufficient conditions (Theorem 2) that a set of equations

$$(1) \quad \sum_{i,j=1}^n d_{kij} \frac{\partial y_i}{\partial x_j} = 0 \quad (k = 1, 2, \dots, (n^2 - n)),$$

where the d_{kij} are constants in a field F , determine a linear associative commutative algebra A over F for which equations (1) are the generalized Cauchy-Riemann differential equations. This will enable us to find solutions of such a set by means of power series in the algebra.

Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be a proper basis for a linear associative commutative algebra A with an identity over the field F . Multiplication will be defined by

$$(2) \quad \epsilon_i \epsilon_j = \sum_{k=1}^n C_{ijk} \epsilon_k \quad (i, j = 1, 2, \dots, n)$$

where the C_{ijk} are in F . Denote by R_i the matrix (C_{isr}) where r is the row index and s the column index. If $\alpha = a_1 \epsilon_1 + a_2 \epsilon_2 + \dots + a_n \epsilon_n$ is any element of A , then $\alpha \mapsto a_1 R_1 + a_2 R_2 + \dots + a_n R_n$ is an isomorphism known as the first regular representation of A by matrices.

Let U denote a system of functions $y_i(x_1, x_2, \dots, x_n)$ of n variables $x_1, x_2, \dots, x_n$ of F , and let the y_i be analytic in a simply-connected region R of n -space. Then $\eta = \sum_{i=1}^n y_i \epsilon_i$ will be called a function over A of the variable $\xi = \sum_{i=1}^n x_i \epsilon_i$.

Since the algebra is commutative we define with Wagner [2, p. 456] that $\eta = \sum_{i=1}^n y_i \epsilon_i$ be an analytic function of ξ if the y_i are in U and the Jacobian matrix $(\partial y_r / \partial x_s)$ is in A , that is, if there exist functions $z_1, z_2, \dots, z_n$ such that

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$$(3) \quad \left(\frac{\partial y_r}{\partial x_s} \right) = z_1 R_1 + z_2 R_2 + \cdots + z_n R_n.$$

This implies a set of homogeneous relations among the $\partial y_r / \partial x_s$ called the generalized Cauchy-Riemann differential equations of the algebra with basis $\epsilon_1, \epsilon_2, \dots, \epsilon_n$.

LEMMA. *If A is a linear associative commutative algebra of order n with an identity over F , then a necessary condition that $\eta = \sum_{i=1}^n y_i \epsilon_i$ be analytic is that the components of η satisfy a linearly independent system of differential equations*

$$(4) \quad \sum_{i,j=1}^n d_{kij} \frac{\partial y_i}{\partial x_j} = 0 \quad (k = 1, 2, \dots, (n^2 - n)),$$

where the d_{kij} are constants in F depending only on the multiplication constants C_{ikj} of the algebra, and such that

$$(5) \quad \sum_{i=1}^n d_{kii} = 0 \quad (k = 1, 2, \dots, (n^2 - n)).$$

By the definition of derivative the matrix equation

$$(6) \quad u_1 R_1 + u_2 R_2 + \cdots + u_n R_n = \left(\frac{\partial y_r}{\partial x_s} \right)$$

gives n^2 equations in the n unknowns, with coefficients among the set C_{ijk} . From any n of the equations (6) for which the determinant is not zero, we can solve for the u_i , and substitution in the remaining equations leads to $n^2 - n$ equations of the form (4). Under a change of basis

$$(7) \quad \epsilon_i = \sum_{j=1}^n t_{ij} \epsilon'_j, \quad t_{ij} \in F \quad (i = 1, 2, \dots, n),$$

we have $x'_j = \sum_{i=1}^n t_{ij} x_i$ and $y'_j = \sum_{i=1}^n t_{ij} y_i$. Therefore

$$\sum_{i,j=1}^n d_{kij} \frac{\partial y_i}{\partial x_j} = \sum_{u,v=1}^n d'_{kuv} \frac{\partial y'_u}{\partial x'_v}.$$

Let $D_k = (d_{krs})$ and $D'_k = (d'_{krs})$; then

$$(8) \quad D'_k = T^{-1} D_k T$$

where $T = (t_{rs})$. Since the trace of a matrix is invariant under similarity transformation, we see that $\sum_{i=1}^n d_{kii}$ is invariant under change

of basis of the algebra A .

Wagner [2, equations 3] has shown that if ϵ_1 is the identity, the generalized Cauchy-Riemann differential equations may be written

$$(9) \quad -\frac{\partial y_r}{\partial x_s} + \sum_{i=1}^n C_{isr} \frac{\partial y_i}{\partial x_1} = 0 \quad (r, s = 1, 2, \dots, n).$$

Since ϵ_1 is the identity, $C_{1sr} = \delta_{sr}$ (Kronecker δ). For $s=1$ we get n equations

$$-\frac{\partial y_r}{\partial x_1} + \sum_{i=1}^n C_{1ir} \frac{\partial y_i}{\partial x_1} = 0 \quad (r = 1, 2, \dots, n)$$

which are identically zero. The $n^2 - n$ remaining equations are in the form (4). If, in (9), $r=s \neq 1$ we get

$$-\frac{\partial y_s}{\partial x_s} + \frac{\partial y_1}{\partial x_1} + \sum_{i=2}^n C_{iss} \frac{\partial y_i}{\partial x_1}$$

because $C_{1ss} = 1$, hence $\sum_{i=1}^n d_{kii} = -1 + 1 = 0$. If $r \neq s$, $s \neq 1$, $C_{1sr} = 0$ and the equations reduce to

$$-\frac{\partial y_r}{\partial x_s} + \sum_{i=2}^n C_{isr} \frac{\partial y_i}{\partial x_1} = 0$$

so that $d_{kii} = 0$ for every i . Hence in every case we have

$$(10) \quad \sum_{i=1}^n d_{kii} = 0$$

and the lemma is proved.

THEOREM 1. If $A_i = (a_{isr})$ ($i = 1, 2, \dots, n$) is a set of n by n matrices over F such that

$$(11) \quad A_i A_j = A_j A_i \quad (i, j = 1, 2, \dots, n)$$

and if there is a p such that

$$(12) \quad a_{ipr} = \delta_{ir} \quad (i, r = 1, 2, \dots, n),$$

then

$$(13) \quad A_i A_j = \sum_{t=1}^n a_{ijt} A_t \quad (i, j = 1, 2, \dots, n)$$

and the A_i form a basis for a linear, commutative, associative algebra of order n over F with an identity element.

By (12) the element of u th row and p th column of $A_i A_j$ is

$$\sum_{t=1}^n a_{itu} a_{jpt} = \sum_{t=1}^n a_{itu} \delta_{jt} = a_{iju}.$$

By change of notation we obtain the element of the u th row and the p th column of $A_j A_i$ to be a_{jiu} . Thus we have

$$(14) \quad a_{iju} = a_{jiu} \quad (i, j, u = 1, 2, \dots, n).$$

Another form of (11) is

$$(15) \quad \sum_{t=1}^n a_{itu} a_{jvt} = \sum_{t=1}^n a_{jtu} a_{ivt}.$$

Thus we have

$$\begin{aligned} A_i A_j &= \left(\sum_{t=1}^n a_{itr} a_{jst} \right) = \left(\sum_{t=1}^n a_{jtr} a_{ist} \right) && \text{(by (15))} \\ &= \left(\sum_{t=1}^n a_{jtr} a_{sit} \right) && \text{(by (14))} \\ &= \left(\sum_{t=1}^n a_{sit} a_{jrt} \right) && \text{(by (15))} \\ &= \left(\sum_{t=1}^n a_{itr} a_{ijt} \right) && \text{(by (14))} \\ &= \sum_{t=1}^n a_{ijt} A_t. \end{aligned}$$

From (12) we see that the A_i are linearly independent with respect to F . Equations (12) and (14) together show that $A_p = I$. Hence the A_i form a basis for an algebra A whose constants of multiplication are $C_{ijk} = a_{ijk}$, so that $R_i = A_i$.

Let

$$(16) \quad f_k \equiv \sum_{i,j=1}^n d_{kij} \frac{\partial y_i}{\partial x_j} \quad (k = 1, 2, \dots, (n^2 - n))$$

such that d_{kij} are in a field F and the f_k are linearly independent with respect to F . If there is a p such that it is possible to solve for each $\partial y_i / \partial x_j$ in the system

$$(17) \quad \sum_{i,j=1}^n d_{kij} \frac{\partial y_i}{\partial x_j} = 0 \quad (k = 1, 2, \dots, (n^2 - n))$$

as a linear function of $\partial y_1/\partial x_p, \partial y_2/\partial x_p, \dots, \partial y_n/\partial x_p$, then (17) may be put in a matrix form

$$(18) \quad \left(\frac{\partial y_r}{\partial x_s} \right) = \left(\sum_{t=1}^n a_{tsr} \frac{\partial y_t}{\partial x_p} \right) = \sum_{t=1}^n (a_{tsr}) \frac{\partial y_t}{\partial x_p} = \sum_{t=1}^n A_t \frac{\partial y_t}{\partial x_p}$$

by adjunction of the n identities

$$\frac{\partial y_i}{\partial x_p} = \frac{\partial y_i}{\partial x_p} \quad (i = 1, 2, \dots, n).$$

THEOREM 2. *Suppose the system of differential equations (17) has the property that for some fixed integer p , it implies the set (18). Suppose further that the matrices $A_i = (a_{isr})$, $i = 1, 2, \dots, n$, are commutative and*

$$(19) \quad \sum_{i=1}^n d_{kii} = 0 \quad (k = 1, 2, \dots, (n^2 - n)).$$

Then there is a uniquely determined linear, commutative associative algebra A , over F , for which (17) is a set of generalized Cauchy-Riemann differential equations (in the sense of the lemma).

Since each equation of (18) which is not an identity is a linear combination of the set (17), it follows from (19) that $a_{tpr} = \delta_{rt}$. Hence by Theorem 1 the A_i ($i = 1, 2, \dots, n$) form a basis for an algebra A with $A_p = I$ as the identity element. It is seen that (17) and hence (16) is a set of generalized Cauchy-Riemann differential equations for the algebra.

For $n = 2$ we have the

COROLLARY. *A necessary and sufficient condition that the linearly independent equations*

$$(20) \quad \sum_{i,j=1}^2 d_{kij} \frac{\partial y_i}{\partial x_j} = 0 \quad (k = 1, 2)$$

determine an algebra A for which (20) is a set of generalized Cauchy-Riemann differential equations is that

$$(21) \quad d_{k11} + d_{k22} = 0 \quad (k = 1, 2).$$

The necessity of (21) follows from the lemma.

Since (21) holds and the matrix of the coefficients of (20) is of rank 2, (20) may be put into form (18). Therefore either $A_1 = I$ and therefore is commutative with A_2 or $A_2 = I$ and is commutative with A_1 . Therefore the corollary follows from Theorem 2.

Ketchum [3, pp. 646 and 653] proved that every solution $y_1, y_2, \dots, y_n$ of the generalized Cauchy-Riemann differential equations for a commutative algebra may be expressed in the form

$$\eta = \sum_{i=1}^n y_i \epsilon_i = \sum_{j=0}^{\infty} a_j \xi^j$$

where $\xi = \sum_{i=1}^n x_i \epsilon_i$. Therefore if equations (1) determine an algebra, the solutions of the equations may be obtained by use of power series in the algebra.

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