SUPPORT, CONVERGENCE, AND DIFFERENTIABILITY PROPERTIES OF GENERALIZED CONVEX FUNCTIONS

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1. Introduction. The notion of sub- F functions has been introduced by Beckenbach [1] and studied further by Peixoto [5; 6; 7] and Bonsall [3]. Briefly, these may be described as follows: In an interval $I: a < x < b$, there exists a two parameter family $\mathcal{F}$ of continuous functions F such that given (x_1, y_1) and (x_2, y_2) , with x_1 and x_2 in I and $x_1 \neq x_2$, there is exactly one function F of $\mathcal{F}$ such that $F(x_1) = y_1$ and $F(x_2) = y_2$. A function f is sub- F provided if f and F agree at x_1 and x_2 then $f \leq F$ between x_1 and x_2 .

Beckenbach showed that the sub- F functions possess a number of properties analogous to those of convex functions, and we list below such of his results as will be used later in this note. It will be convenient to speak of the functions of $\mathcal{F}$ as F -lines, and it is clear what is meant by saying that two points determine an F -line. All x appearing will belong to I , although it may not be so stated in each case. The following statements are given informally, but their exact content is easy to see.

B1. *If two F -lines have a point in common and are not identical, they cross at that point.*

B2. *If two sequences of points tend to limits with different abscissas, the F -lines determined by the successive pairs of points of the sequences tend to the F -line determined by the limit points, uniformly in every compact subinterval of I .*

B3. *If f is sub- F and agrees with F at x_1 and x_2 , then outside the interval (x_1, x_2) , $f \geq F$.*

B4. *A sub- F function is continuous.*

An F -line is a support of f at x_0 provided $F \leq f$ in I and $F(x_0) = f(x_0)$. Peixoto [5] obtained among others the following results.

P1. *A sub- F function has a support at each point x_0 . If the support is not unique, there are two extreme supports, F^+ and F^- , crossing at x_0 and such that (a) every support to f at x_0 lies between F^+ and F^- , and (b) every F -line which lies between F^+ and F^- supports f at x_0 .*

The support F^+ is obtained by passing the F -line through $(x_0, f(x_0))$ and $(x_1, f(x_1))$, where $x_1 > x_0$, and allowing x_1 to approach x_0 .

P2. *If almost everywhere in I each F -line has finite lower derivatives, then a sub- F function has a derivative almost everywhere.*

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P3. *If all F passing through (x_0, y_0) have a uniformly bounded second derivative in a neighborhood of x_0 , then a sub- F function has a derivative except at countably many points.*

In the present note we shall obtain further supporting and differentiability properties partly overlapping and partly extending those obtained by Peixoto. The author has had occasion to make analytic use of a certain class of sub- F functions, namely the sub-sine functions (see §4), and the hypotheses placed on $\mathcal{F}$ in this note are suggested by that use. We shall also obtain a pair of theorems on the convergence of sequences of sub- F functions.

The families considered by Bonsall are somewhat less general than those of the present paper, being the solutions of certain second order differential equations. Also he obtains a stronger result concerning the one item where there is essential overlap in subject matter between the two papers; namely, he shows that his sub-functions have second derivatives almost everywhere. Our Theorem 5 is related to his corollary to Theorem 1, and our Theorem 3 to his Theorem 2. In our application, the family involved is of the type considered by Bonsall, and the result could equally have been deduced from his work.

2. Support and convergence.

LEMMA 1. *Let F_0 and F_1 support the sub- F function f at x_0 and x_1 respectively, where $x_0 < x_1$. Then for $x > x_1$, we have $F_0 \leq F_1$.*

Suppose that for $x_2 > x_1$ we have $F_1(x_2) < F_0(x_2)$. Since $F_1(x_1) = f(x_1) \geq F_0(x_1)$, F_1 and F_0 must cross at x_1 or to the right of it. In either case, to the left of x_1 , $F_1 > F_0$; in particular, $F_1(x_0) > F_0(x_0) = f(x_0)$, which contradicts the fact that F_1 supports f at x_1 .

THEOREM 1. *A sub- F function has a unique support at all but a countable set of points.*

First consider two points x_0 and x_1 at which the supports are not unique, and at which the extreme supports are F_0^+ , F_0^- , and F_1^+ , F_1^- , respectively. Let A_0 be the open set of points to the right of x_0 and between F_0^+ and F_0^- ; similarly define A_1 . Then A_0 and A_1 do not intersect. For suppose that (x_2, y_2) is a common point. The F -line F^* determined by (x_2, y_2) and $(x_0, f(x_0))$ supports f at x_0 , by P1. By Lemma 1, $F^* \leq F_1^-$ for $x > x_1$, and in particular, $F^*(x_2) = y_2 \leq F_1^-(x_2)$. This makes it impossible for (x_2, y_2) to be in A_1 .

To each point x at which the support to f is not unique we assign the open set A to the right of x and between the extreme supports. By

the preceding argument these open sets are disjoint and hence countable. This completes the proof.

THEOREM 2. *Let $\{f_n\}$ be a sequence of sub- F functions converging to a limit function f . Then f is sub- F , and the convergence is uniform in every compact subinterval of I .*

The F -line determined by $(x_i, f(x_i))$ and $(x_j, f(x_j))$ we denote by F_{ij} ; similarly for f_n . Let $x_0 < x_1 < x_2$. Then $f_n(x_1) \leq F_{n,02}(x_1)$. By B2, $\lim F_{n,02} = F_{02}$; thus $\lim f_n(x_1) = f(x_1) \leq \lim F_{n,02}(x_1) = F_{02}(x_1)$, and f is sub- F .

Now suppose that the convergence is not uniform. There exists a sequence $\{x_n\}$ tending to a point x_0 of I such that $|f_n(x_n) - f(x_n)| > \epsilon > 0$. We may assume that $x_n > x_0$ without loss of generality. By choosing suitable subsequences, we can assume that either (a) $f_n(x_n) < f(x_n) - \epsilon$, or (b) $f_n(x_n) > f(x_n) + \epsilon$. First take case (a). Let $x^* < x_0$ and F^* be the F -line determined by $(x^*, f(x^*) + \epsilon/2)$ and $(x_0, f(x_0) - \epsilon/2)$. The functions f and F^* are continuous, and by choosing n sufficiently large, we can assure that $|f(x_n) - f(x_0)| < \epsilon/4$ and also that $|F^*(x_n) - F^*(x_0)| < \epsilon/4$. Then

$$f_n(x_n) < f(x_n) - \epsilon < f(x_0) - 3\epsilon/4 = F^*(x_0 - \epsilon/4 < F^*(x_n).$$

Also let n be so great that $f_n(x^*) < f(x^*) + \epsilon/2 = F^*(x^*)$. Then $F^* > f_n$ at x^* and x_n , and since f is sub- F , we have $f_n(x_0) < F^*(x_0)$. Letting $n \rightarrow \infty$, it follows that $f(x_0) = \lim f_n(x_0) \leq F^*(x_0) = f(x_0) - \epsilon/2$, an impossibility. Thus (a) is not possible. If case (b) holds, let $x^* > x_0$ and F_n^* be determined by f_n at x_0 and x^* , and F^* by f at the same points. Then $f_n(x_n) \leq F_n^*(x_n)$. Now $F_n^*(x) \rightarrow F^*(x)$ uniformly in an interval containing x_0 , and so $F_n^*(x_n) \rightarrow F^*(x_0) = f(x_0)$. This says that $\limsup f_n(x_n) \leq f(x_0)$, which contradicts (b), and the theorem is proved.

3. Differentiability. Beckenbach points out that the function F is itself sub- F and constructs a family $\mathcal{F}$ all of whose members are nowhere differentiable. A sub- F function thus need not be differentiable unless further conditions are placed on $\mathcal{F}$. We now place two such conditions.

A. *Each F -line shall be continuously differentiable.*

B. *If $F_n \rightarrow F$, then $F'_n \rightarrow F'$, uniformly in every compact subinterval of I .*

Henceforth $\mathcal{F}$ will be assumed to satisfy A and B, and the sub- F functions will be said to be sub- FS .

LEMMA 2. *Let J be a compact subinterval of I , and δ and M two*

positive numbers. There exists a function $B(J, \delta, M)$ such that if x, x_1 , and x_2 are in J , $|x_1 - x_2| > \delta$, $|y_1| < M$, $|y_2| < M$, then $|F'_{12}(x)| < B(J, \delta, M)$.

The proof follows straightforwardly from A and B and simple compactness considerations, and will be omitted. One might imagine that B follows from A much in the same manner that Beckenbach was able to prove B2 from the continuity and uniqueness of the F . This is not the case, however. It is easy to give examples to show that even A and Lemma 2 do not imply B; in fact, no amount of smoothness of the individual curves will imply B. Theorems 3 and 4 below are proved on the basis of A and Lemma 2, but the other results also use B.

THEOREM 3. *A sub-FS function satisfies a Lipschitz condition in every compact subinterval J of I , and thus is absolutely continuous and has a derivative almost everywhere, which is bounded in J .*

If J is the interval $a' \leq x \leq b'$, we consider the interval $J_1: a' - \delta \leq x \leq b' + \delta$, where $\delta > 0$ is sufficiently small so that $J_1 \subset I$. Let M be the maximum of $|f|$ in J_1 , and let F^+ be determined by $(x_0, f(x_0))$ and $(b' + \delta, f(b' + \delta))$ and F^- by $(x_0, f(x_0))$ and $(a' - \delta, f(a' - \delta))$. If $x_1 \in J$, we see from B3 that

$$\begin{aligned} F'^-(\bar{x}_1) &= \frac{F^-(x_1) - F^-(x_0)}{x_1 - x_0} \leq \frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq \frac{F^+(x_1) - F^+(x_0)}{x_1 - x_0} \\ &= F'^+(\bar{x}_1^*), \end{aligned}$$

where $\bar{x}_1$ and x_1^* are between x_0 and x_1 , and hence are in J . By Lemma 2 it follows that $|\Delta f / \Delta x| < B(J_1, \delta, M)$, and the theorem is proved.

THEOREM 4. *If $\lim f_n = f$, where f_n is sub-FS, then the f_n satisfy a uniform Lipschitz condition in any compact subinterval of I , and thus the derivatives f'_n are uniformly bounded.*

The proof is just like that of Theorem 3, with the added remark that the f_n have a common bound.

THEOREM 5. *If f is sub-FS and has a unique support F_0 at x_0 , then $f'(x_0)$ exists and equals $F'_0(x_0)$.*

Let $x_1 > x_0$, and consider $F_{01}(x)$. We have

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{F_{01}(x_1) - F_{01}(x_0)}{x_1 - x_0} = F'_{01}(\bar{x}_1),$$

where $x_0 < \bar{x}_1 < x_1$. As $x_1 \rightarrow x_0$, we see as in P1 that $F_{01}(x)$ tends to

$F_0(x)$, the unique support to f at x_0 . By hypothesis B, $F'_{01}(x) \rightarrow F'_0(x)$, uniformly in an interval containing x_0 . Since $x_1 \rightarrow x_0$, we have that $F'_{01}(x_1) \rightarrow F'_0(x_0)$, and f has its right-hand derivative equal to $F'_0(x_0)$. Similarly for the left-hand derivative and the theorem is proved. Actually one sees that we have proved more than the theorem stated; namely that the right- and left-hand derivatives always exist and equal the slopes of the extreme supports at x_0 .

The converse of Theorem 5 is not true: f' may exist without there being a unique support. An example of this is given by the family $\mathcal{F}: y = (ax+b)^3$. The function $|x|^3$ is sub-FS and has a derivative and many supports at the origin.

THEOREM 6. *A sub-FS function has a derivative at all but a countable set of points.*

This is a consequence of Theorems 1 and 5.

REMARK. The hypotheses under which we showed that f has a derivative almost everywhere are more restrictive than Peixoto's, but the result is stronger. Our hypotheses for proving Theorem 6 neither imply nor are implied by those of Peixoto for P3. On the other hand they are such as to permit us to show the connection between absence of derivative and non-unique support.

THEOREM 7. *Let $\{f_n\}$ be a sequence of sub-FS functions with limit f . Then with the exception of a countable set, $\lim f'_n(x) = f'(x)$.*

Let x_0 be a point at which f and all f_n have unique supports. There are only countably many exceptions. Let $x_1 > x_0$ and let $F_{n,0}$ be the support to f_n at x_0 . Then for $x > x_0$, $F_{n,01}(x) \geq F_{n,0}(x)$, and so $F'_{n,01}(x_0) \geq F'_{n,0}(x_0) = f'_n(x_0)$. Operating similarly with $x_2 < x_0$, we find

$$F'_{n,20}(x_0) \leq f'_n(x_0) \leq F'_{n,01}(x_0).$$

As $n \rightarrow \infty$, $F_{n,20} \rightarrow F_{20}$ and $F_{n,01} \rightarrow F_{01}$, and by B,

$$F'_{20}(x_0) \leq \liminf f'_n(x_0) \leq \limsup f'_n(x_0) \leq F'_{01}(x_0).$$

Now let $x_1 \rightarrow x_0$ and $x_2 \rightarrow x_0$. As in P1, $F_{20} \rightarrow F_0$ and $F_{01} \rightarrow F_0$, and by B, $F'_{20}(x_0) \rightarrow F'_0(x_0)$, $F'_{01}(x_0) \rightarrow F'_0(x_0)$. Thus $\lim f'_n(x_0) = F'_0(x_0) = f'(x_0)$.

4. An application. A function is called sub-sine if in every interval of length less than 2π it is sub- F , with $\mathcal{F}$ the family: $a \cos \theta + b \sin \theta$. $\mathcal{F}$ clearly satisfies conditions A and B. The sub-sine functions $p(\theta)$ with period 2π are the supporting functions of convex curves [4]. If p is sufficiently regular, the formula for the area S of the convex curve represented by p is $S = (1/2) \int_0^{2\pi} (p^2 - p'^2) d\theta$ [2, p. 58]. Any

supporting function p is the limit of analytic ones, p_n [2, p. 36]. Because of the periodicity of p we see by Theorems 2, 4, and 7 that $p_n \rightarrow p$ uniformly and $p'_n \rightarrow p'$ boundedly for all θ ; thus $S = \lim S_n = \lim (1/2) \int_0^{2\pi} (p_n^2 - p_n'^2) d\theta = (1/2) \int_0^{2\pi} (p^2 - p'^2) d\theta$, and we have a simple proof of the known fact that the integral $(1/2) \int_0^{2\pi} (p^2 - p'^2) d\theta$ represents the area for the general convex curve.

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