A CARDINAL NUMBER ASSOCIATED WITH A FAMILY OF SETS[1]

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Let U be a family of nonempty subsets of an abstract set R, partially ordered by set inclusion.[2] The smallest cardinal number which is the power of the union of a maximal family of incomparable elements of U shall be defined as the "maximal density" of U (md (U)). The smallest cardinal number which is the md (V) of some coinitial subfamily V of U shall be defined as the "containing maximal density" of U (cmd (U)). The principal result of this paper is Theorem 2, which states that $\prod_{\xi<\alpha}$ cmd $(U^\xi)=$cmd $(\prod_{\xi<\alpha} U^\xi)$.

Before turning to our main result we consider the containing maximal density of a ramified family of sets.[3]

THEOREM 1. *If U is a ramified family of sets, then the maximal density of U equals the containing maximal density of U.*

PROOF. Let $V=\{E\}$ be a coinitial subfamily of $U=\{D\}$, such that md $(V)=$cmd (U). Let $M=\{H\}$ be a maximal family of incomparable elements of V for which md $(V)=p(M)$.[4] We shall now show that M is a maximal family of incomparable elements of U. Let D be any element in U, and E any element of V which is a subset of D. The element E certainly exists since V is a coinitial subset of U. The family M being a maximal family of incomparable elements of V, there exists an element H in M which is comparable with E. If H is a subset of E, then H is also a subset of D. Suppose that E is a subset of H. Since U is ramified, it follows that the two elements D and H are comparable. Consequently each element of U is comparable with some element in M. Thus M is a maximal family of incomparable elements in U. Therefore

$$\text{md }(U) \leqq \text{md }(V) = \text{cmd }(U) \leqq \text{md }(U).$$

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[2] In the sequel U and U^ξ will be a family of subsets of the sets R and R^ξ respectively.

[3] The family of sets U is a ramified family of sets if, for each element E in U, the family of sets $\{D \mid D\supseteq E, D\in U\}$ is monotone.

[4] Let Y be a family of sets. By $p(Y)$ is meant the power of the set which is the set union of the elements in Y. If Y consists of only one set, say A, then $p(Y)$ is the power of A. In this case, $p(A)$ is also used, i.e., $p(A)=p(Y)$.

Thus $\mathrm{md}(U) = \mathrm{cmd}\ (U)$.

In preparation for Theorem 2, two lemmas are needed.

LEMMA 1. *The containing maximal density of U is the smallest cardinal number equal to $p(V)$, for some coinitial subfamily V of U.*

PROOF. Let $V = \{E\}$ be any coinitial subfamily of U for which $\mathrm{cmd}\ (U) = \mathrm{md}\ (V)$. Let $M = \{E_\nu\}$ be a maximal family of incomparable elements of V such that $p(M) = \mathrm{md}\ (V)$. Now the subfamily of V,

$$Y = \{E \mid E \subseteq E_\nu,\ E \in V,\ E_\nu \in M, \text{ for some } \nu\},$$

is coinitial in V. Thus $p(Y) \leqq \mathrm{md}\ (V) = \mathrm{cmd}\ (U)$. Now let $p(Z)$ be the smallest cardinal number for some coinitial subfamily Z of U. Clearly $\mathrm{cmd}\ (U) \leqq \mathrm{md}\ (Z) \leqq p(Z)$. This completes the proof.

LEMMA 2. *To each family $U = \{D\}$, there corresponds a coinitial subfamily $V = \{E\}$, and a subfamily, $Y = \{D_\nu \mid \nu < \delta\}$ of V, which have the following properties:*

(1) *each element of V is p-homogeneous;*[5]

(2) $\{E \mid E \subseteq D_\xi,\ E \in V\} \cap \{E \mid E \subseteq D_\nu,\ E \in V\} = \emptyset$ *for* $\xi \neq \nu$;

(3) *if $\{G_\xi \mid \xi < \delta\}$ is any subfamily of V in which G_ξ is a subset of D_ξ for each ξ, then*

$$p\left(G_\nu - \bigcup_{\xi<\nu} D_\xi\right) = p\left(D_\nu - \bigcup_{\xi<\nu} D_\xi\right) \qquad (\nu < \delta),$$

(4) $p[\bigcup_{\nu<\delta} (D_\nu - \bigcup_{\xi<\nu} D_\xi)] \geqq \mathrm{cmd}\ (V)$.

PROOF. If V is a coinitial subfamily of U such that $\mathrm{md}\ (V) = \mathrm{cmd}\ (U)$, and

$$Z = \{E \mid E \in V,\ E \text{ is } p\text{-homogeneous}\},$$

then Z is a coinitial subfamily of U for which $\mathrm{md}\ (Z) = \mathrm{cmd}\ (U)$. In order to simplify the notation, it is assumed that U has the two properties of Z, i.e., (a) $\mathrm{md}\ (U) = \mathrm{cmd}\ (U)$, and (*b*) each element D in U is p-homogeneous. Well order the elements of U, D_0 being the first element. Suppose that the family of sets $\{D_\xi \mid \xi < \lambda\}$ has already been defined. Denote by D_λ the first element D_* in U which satisfies the following two conditions:

(c) if D is a subset of D_*, where D is in U, then D is not a subset of $\bigcup_{\xi<\lambda} D_\xi$;

[5] An element E in V is p-homogeneous if $p(B) = p(E)$ for each element B in V which is a subset of E. See Erdös and Tarski, *On families of mutually exclusive sets*, Ann. of Math. vol. 44 (1943) pp. 315–329.

(d) if D is a subset of D_*, where D is in U, then $p(D-\bigcup_{\xi<\lambda} D_\xi) = p(D_* - \bigcup_{\xi<\lambda} D_\xi)$. If the set of elements satisfying (c) is nonempty, then the element D_λ certainly exists. Let $\{D_\xi \mid \xi<\delta\}$ be a maximal family obtained in this way.

Let $V=\{E\}$ be the subfamily of U,

$$V = \left\{D \mid D \subseteq \bigcup_{\xi<\delta} D_\xi,\ D \in U\right\}.$$

Conditions (1), (2), (3), and (4) are automatically satisfied. It is necessary to show only that V is a coinitial subfamily of U. Suppose that D_* is an element of U which contains no element of V. Then the element D_* satisfies condition (c). This implies that the family $\{D_\xi \mid \xi<\delta\}$ is not maximal. From this contradiction we obtain our conclusion.

We now prove our main result.

THEOREM 2. cmd $(\prod_{\xi<\alpha} U^\xi) = \prod_{\xi<\alpha}$ cmd (U^ξ), *where by* $\prod_{\xi<\alpha} U^\xi$ *is meant the cartesian product of the* U^ξ.[6]

PROOF. To each family U^ξ, associate a family of sets, $Y^\xi = \{D^\xi_\gamma \mid \gamma<\delta_\xi\}$, which satisfies the conclusions of Lemma 2. Let U be a coinitial subfamily of $\prod_\xi U^\xi$ for which $p(U) = \mathrm{md}\,(U) = \mathrm{cmd}\,(U)$. In the following, β, γ, and ξ denote indices that range over specified sets of ordinal numbers, whereas μ and ν denote indices that range over the class of all α-sequences $\sigma=\{\sigma_\xi\}$, $\xi<\alpha$, where $\sigma_\xi<\delta_\xi$ for each ξ. The notation $\mu<\nu$ refers to the partial ordering defined by

$$\mu < \nu \text{ if and only if } \begin{cases} \mu_\xi \leqq \nu_\xi & \text{for all } \xi < \alpha, \text{ and} \\ \mu_\xi < \nu_\xi & \text{for some } \xi < \alpha. \end{cases}$$

For each ν, $\prod_\xi D^\xi_{\nu_\xi}$ belongs to $\prod_\xi U^\xi$. Since U is a coinitial subfamily of $\prod_\xi U^\xi$, there exists an element D_ν in U such that $D_\nu \subseteq \prod_\xi D^\xi_{\nu_\xi}$. D_ν is of the form $D_\nu = \prod_\xi E^\xi_\nu$, where for each $\xi<\alpha$, E^ξ_ν is an element of U^ξ and $E^\xi_\nu \subseteq D^\xi_{\nu_\xi}$. It now follows that

$$\mathrm{cmd}\left(\prod_\xi U^\xi\right) = p(U) \geqq p\left(\bigcup_\nu D_\nu\right)$$
$$\geqq p\left[\bigcup_\nu \left(D_\nu - \bigcup_{\mu<\nu} D_\nu\right)\right] \geqq p\left[\bigcup_\nu \left(D_\nu - \bigcup_{\mu<\nu} \prod_\xi D^\xi_{\mu_\xi}\right)\right].$$

Since each two terms inside the last bracket are disjoint,

[6] The cartesian product of the U^ξ is the family of sets $\{H \mid H = D^0 \times D^1 \times \cdots,\ D^\xi \in U^\xi\}$.

$$p\left[\bigcup_{\nu}\left(D_{\nu}-\bigcup_{\mu<\nu}\prod_{\xi}D^{\xi}_{\mu_{\xi}}\right)\right]$$
$$=\sum_{\nu}p\left(D_{\nu}-\bigcup_{\mu<\nu}\prod_{\xi}D^{\xi}_{\mu_{\xi}}\right)$$
$$=\sum_{\nu}p\left(\prod_{\xi}E^{\xi}_{\nu}-\bigcup_{\mu<\nu}\prod_{\xi}D^{\xi}_{\mu_{\xi}}\right),$$

which, since $\prod_{\xi}(E^{\xi}_{\nu}-\bigcup_{\beta<\nu_{\xi}}D^{\xi}_{\beta})\subseteq(\prod_{\xi}E^{\xi}_{\nu}-\bigcup_{\mu<\nu}\prod_{\xi}D^{\xi}_{\mu_{\xi}})$, is

$$\geqq\sum_{\nu}p\left[\prod_{\xi}\left(E^{\xi}_{\nu}-\bigcup_{\beta<\nu_{\xi}}D^{\xi}_{\beta}\right)\right]$$
$$=\sum_{\nu}\prod_{\xi}p\left(E^{\xi}_{\nu}-\bigcup_{\beta<\nu_{\xi}}D^{\xi}_{\beta}\right)\Bigg]$$
$$=\sum_{\nu}\prod_{\xi}p\left(D^{\xi}_{\nu_{\xi}}-\bigcup_{\beta<\nu_{\xi}}D^{\xi}_{\beta}\right) \qquad \text{(by condition 3)}$$
$$=\prod_{\xi}\sum_{\gamma<\delta_{\xi}}p\left(D^{\xi}_{\gamma}-\bigcup_{\beta<\gamma}D^{\xi}_{\beta}\right) \qquad \text{(by the distributive law)}$$
$$=\prod_{\xi}p\left[\bigcup_{\gamma<\delta_{\xi}}\left(D^{\xi}_{\gamma}-\bigcup_{\beta<\gamma}D^{\xi}_{\beta}\right)\right]$$
$$\geqq\prod_{\xi}\operatorname{cmd}(U^{\xi}).$$

Thus cmd $(\prod_{\xi}U^{\xi})\geqq\prod_{\xi}$ cmd (U^{ξ}). Since the reverse inequality is obviously true, it follows that $\prod_{\xi}$ cmd $(U^{\xi})=$ cmd $(\prod_{\xi}U^{\xi})$.

For each $\xi<\mu$ let U^{ξ} be a family of infinite subsets of a set R^{ξ}. By $\prod'_{\xi<\mu}U^{\xi}$ we shall mean the family of sets

$\{G\mid G=\prod H^{\xi}$, where $H^{\xi}\in U^{\xi}$, or $H^{\xi}=R^{\xi}$, and all but a finite number of the H^{ξ} are $R^{\xi}\}$.

THEOREM 3. *cmd* $(\prod'_{\xi}U^{\xi})$ *is the smallest cardinal number, call it* $\aleph_{\beta}$, *which is in the set of cardinal numbers* $\{\aleph_{\alpha}\mid\aleph_{\alpha}=\text{cmd}\,(U^{\alpha(0)}\times U^{\alpha(1)}\times\cdots\times U^{\alpha(n)})\cdot\prod_{\xi\neq\alpha(j),j\leqq n}p(R^{\xi});\ n<\omega\}$.

PROOF. If V^{ξ} is a coinitial subfamily of U^{ξ}, then cmd $(\prod'U^{\xi})=$ cmd $(\prod'V^{\xi})$. Thus, no generality is lost in assuming that for each ξ

$$p(U^{\xi})=\text{md}\,(U^{\xi})=\text{cmd}\,(U^{\xi}).$$

Furthermore, it may be assumed that

$$\aleph_\beta = \text{cmd } (U^0 \times \cdots \times U^n) \cdot \prod_{\xi > n} p(R^\xi).$$

To prove the theorem, it is sufficient to show that $\aleph_\beta \leqq \text{cmd } (\prod' U^\xi)$. Let U be a coinitial subfamily of $\prod' U^\xi$ for which $p(U) = \text{md } (U) = \text{cmd } (\prod' U^\xi)$, and let D be any element of U. To simplify the notation, suppose that

$$D = D^0 \times \cdots \times D^{n+m} \times \prod_{\xi > n+m} R^\xi.$$

Then

$$\begin{aligned} \text{cmd } (\textstyle\prod' U^\xi) &= \text{md } (U) = p(U) \geqq p(D) \\ &= p(D^0 \times D^1 \times \cdots \times D^{n+m}) \cdot \prod_{\xi > n+m} p(R^\xi) \\ &\geqq \prod_{\xi > n+m} p(R^\xi). \end{aligned}$$

Also,

$$\begin{aligned} \text{cmd } (\textstyle\prod' U^\xi) &\geqq \text{cmd } (U^0 \times \cdots \times U^{n+m}) \\ &= \prod_{\xi \leqq n+m} \text{cmd } (U^\xi) \qquad \text{(Theorem 2)} \\ &= \prod_{\xi \leqq n+m} p(U^\xi) = p\Big(\prod_{\xi \leqq n+m} U^\xi \Big). \end{aligned}$$

Combining our results we get

$$\begin{aligned} \aleph_\beta &\leqq p\Big(\prod_{j \leqq n+m} U^j \Big) \cdot p\Big(\prod_{\xi > n+m} R^\xi \Big) \\ &\leqq \text{cmd } \Big(\prod_\xi{}' U^\xi \Big) \cdot \text{cmd } \Big(\prod_\xi{}' U^\xi \Big) \\ &= \text{cmd } (\textstyle\prod' U^\xi), \end{aligned}$$

the last equality resulting from the fact that as each element of U^ξ is infinite, cmd $(\prod' U^\xi)$ is infinite.

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