

ON THE REPRESENTATION OF AN INTEGER AS THE SUM OF THREE FOURTH POWERS

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1. Introduction. Let $r_{s,k}(N)$ denote the number of representations of a positive integer N as the sum of s k th powers of positive integers. S. Chowla and S. S. Pillai [2] proved that, for infinitely many N ,

$$r_{k,k}(N) > c_1 \log \log N$$

where c_1 (as well as c_2, c_3, \dots in what follows) is a constant independent of N . P. Erdős [5] improved this and proved in an elementary way that, for infinitely many N ,

$$r_{k,k}(N) > \exp \left(c_2 \frac{\log N}{\log \log N} \right).$$

However, for $s < k$ only particular cases have been studied. The only results I have come across are the following two due respectively to S. S. Pillai and S. Chowla (see Chowla [1, p. 122]). For an infinity of N, n ,

$$r_{2,3}(N) > c_3 \log \log N$$

and

$$r_{3,4}(n) > c_4 \frac{\log n}{\log \log n}.$$

In this note (§3) I prove that, for infinitely many N ,

$$r_{3,4}(N) > \exp \left(c_5 \frac{\log N}{\log \log N} \right).$$

The proof is elementary and entirely different from that of Chowla.

2. Lemmas. For the proof of our theorem we require some lemmas. In what follows, all the letters denote positive integers.

LEMMA 1. *Let p_1, p_2, \dots, p_r be the r consecutive primes > 78 and belonging to the arithmetic progression $6n+1$. Then, corresponding to every p_i , there is a constant $k_i < 22464$ such that $n_i^2 = (p_i k_i)^2$ has a representation of the form $x_i^2 + 3y_i^2$ satisfying the conditions that $x_i > 7y_i > 0$ and p_i does not divide $x_i y_i$.*

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PROOF. It is well known (Euler [4]) that every prime of the form $6n+1$ is of the form x^2+3y^2 . Let $p_i = \xi_{i1}^2 + 3\eta_{i1}^2$. The inequalities which follow are easily seen to be strict inequalities.

In case $\xi_{i1} > 2\eta_{i1}$ we write $q_i = p_i$; while in case $\xi_{i1} < 2\eta_{i1}$ we write $q_i = p_i(\lambda^2 + 3)$ where λ is the integer such that

$$(2.1) \quad \frac{\xi_{i1}}{\eta_{i1}} < \lambda < \frac{\xi_{i1}}{\eta_{i1}} + 1 < 3.$$

The number q_i defined in the second case has the representation

$$q_i = \xi_{i2}^2 + 3\eta_{i2}^2 \text{ (say)}$$

where

$$(2.2) \quad \xi_{i2} = \lambda\xi_{i1} + 3\eta_{i1} \quad \text{and} \quad \eta_{i2} = \lambda\eta_{i1} - \xi_{i1}.$$

From (2.1) and (2.2) we get

$$(2.3) \quad \xi_{i2} > 2\eta_{i2}.$$

We see from (2.3) that the representation

$$(2.4) \quad q_i = \xi_{i3}^2 + 3\eta_{i3}^2, \quad 0 < \xi_{i3} < \eta_{i3},$$

is possible, q_i being defined by either $q_i = p_i$ or $q_i = p_i(\lambda^2 + 3)$. Hence

$$(2.5) \quad 16q_i^2 = (6\eta_{i3} - 2\xi_{i3})^2 + 3(2\xi_{i3} + 2\eta_{i3})^2 = \xi_{i4}^2 + 3\eta_{i4}^2 \text{ (say)}.$$

From (2.4) and (2.5) we get

$$0 < \eta_{i4} < \xi_{i4} < 3\eta_{i4}.$$

Now

$$144(\xi_{i4}^2 + 3\eta_{i4}^2) = (6\xi_{i4} + 18\eta_{i4})^2 + 3(6\xi_{i4} - 6\eta_{i4})^2 = \xi_{i5}^2 + 3\eta_{i5}^2 \text{ (say)}.$$

We verify that

$$\xi_{i5} > 3\eta_{i5}.$$

We now consider two cases.

(i) $\xi_{i5} > 7\eta_{i5}$. In this case we write $n_i = 48q_i$.

(ii) $7\eta_{i5} > \xi_{i5} > 3\eta_{i5}$. Now

$$2704(\xi_{i5}^2 + 3\eta_{i5}^2) = (46\xi_{i5} + 42\eta_{i5})^2 + 3(46\eta_{i5} - 14\xi_{i5})^2.$$

We can verify that

$$46\xi_{i5} + 42\eta_{i5} > 7 \mid 46\eta_{i5} - 14\xi_{i5} \mid.$$

In case (ii) we write $n_i = 2496q_i$. The number n_i defined as above for the two cases is such that n_i^2 has a representation $x_i^2 + 3y_i^2$, $x_i > 7y_i > 0$. Using the fact that $p_i > 78$ and does not divide $\xi_{21}\eta_{21}$ we can show that p_i does not divide $x_i y_i$.

LEMMA 2. Let n_i , $i = 1, 2, \dots, r$, be as in Lemma 1. Then the number $n_{j_1}^2 n_{j_2}^2 \dots n_{j_k}^2$, $0 < j_1 < j_2 < \dots < j_k \leq r$, $1 \leq k < r$, has a representation of the form $x^2 + 3y^2$ with $x > 7y > 0$ and such that p_{j_s} , $s = 1, 2, \dots, k$, does not divide xy .

The proof is by induction on k .

LEMMA 3. Let $R_1(N)$ denote the number of representations of an integer N in the form $x^2 + 3y^2$, $x > 7y > 0$. Then, for infinitely many N ,

$$R_1(N) > \exp\left(c_8 \frac{\log N}{\log \log N}\right).$$

PROOF. Let

$$A = n_1 n_2 \dots n_r,$$

$$B = n_{j_1} n_{j_2} \dots n_{j_k}, \quad 0 < j_1 < j_2 < \dots < j_k \leq r, \quad 1 \leq k < r,$$

$$A = BC \quad \text{and} \quad N = A^2,$$

where the n 's are as in Lemma 1.

By Lemma 2, B has a representation $x^2 + 3y^2$, $x > 7y > 0$, such that p_{j_s} , $s = 1, 2, \dots, k$, does not divide xy . This gives a representation $(Cx)^2 + 3(Cy)^2$ for N satisfying the conditions of Lemma 3. No two B 's can give rise to the same representation. For, if B and B' give the same representation we shall have an equality of the type $Cx = C'x'$. If $C \neq C'$ there is a prime p which divides (say) C but not C' . So p divides x' . Also p divides B' since it does not divide C' . This is a contradiction since the representation of B' which we consider is such that if p divides B' then p does not divide x' .

Hence

$$(2.6) \quad R_1(N) \geq \sum_B 1 = 2^r - 2.$$

Now, by the prime number theorem for arithmetic progressions, $p_k < c_7 k \log k$ and

$$\log N = 2(\log n_1 + \dots + \log n_r) < c_8 r \log r,$$

so that

$$(2.7) \quad r > c_9 \frac{\log N}{\log \log N}.$$

From (2.6) and (2.7) the conclusion of the lemma follows.

LEMMA 4. *Let $R_2(N)$ denote the number of representations of an integer N in the form $\xi^2 - \xi\eta + \eta^2$, $\xi > 4\eta > 0$. Then*

$$R_2(N) > \exp\left(c_6 \frac{\log N}{\log \log N}\right)$$

for infinitely many N .

PROOF. Writing $\xi = x + y$ and $\eta = 2y$ and taking N as in Lemma 3, we get

$$R_2(N) = R_1(N) > \exp\left(c_6 \frac{\log N}{\log \log N}\right).$$

3. Theorem. *For infinitely many M ,*

$$r_{3,4}(M) > \exp\left(c_6 \frac{\log M}{\log \log M}\right).$$

PROOF. Consider the identity [3]

$$(k^2 - 2k)^4 + (2k - 1)^4 + (k^2 - 1)^4 = 2(k^2 - k + 1)^4.$$

Writing $k = \xi/\eta$ and multiplying by η^8 this becomes

$$(3.1) \quad (\xi^2 - 2\xi\eta)^4 + (2\xi\eta - \eta^2)^4 + (\xi^2 - \eta^2)^4 = 2(\xi^2 - \xi\eta + \eta^2)^4.$$

If N is as in Lemmas 3 and 4, there are $R_2(N)$ pairs of values of ξ and η , $\xi > 4\eta > 0$, for which the right-hand side of (3.1) can be fixed equal to $M = 2N^4$. Thus we get $R_2(N)$ representations of N in the form $x_1^4 + x_2^4 + x_3^4$.

Now if $\xi > 4\eta$ the greatest and the smallest of the expressions on the left-hand side of (3.1) are respectively $\xi^2 - \eta^2$ and $2\xi\eta - \eta^2$. If ξ_1, η_1 and ξ_2, η_2 give the same representation, then

$$\xi_1^2 - \xi_1\eta_1 + \eta_1^2 = \xi_2^2 - \xi_2\eta_2 + \eta_2^2,$$

$$\xi_1^2 - \eta_1^2 = \xi_2^2 - \eta_2^2,$$

and

$$2\xi_1\eta_1 - \eta_1^2 = 2\xi_2\eta_2 - \eta_2^2.$$

These relations are easily seen to be inconsistent unless $\xi_1 = \xi_2$ and $\eta_1 = \eta_2$. Hence all the representations given by different pairs of values of ξ and η are distinct.

It follows that

$$\begin{aligned} r_{3,4}(M) &\geq R_2(N) \\ &> \exp\left(c_6 \frac{\log N}{\log \log N}\right) \\ &> \exp\left(c_6 \frac{\log M}{\log \log M}\right). \end{aligned}$$

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