ON THE UNIVERSAL COVERING SPACE AND THE FUNDAMENTAL GROUP

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In this note it will be shown that when A is a retract (cf. [3])[1] of X, then the same relationship holds between the universal covering spaces of A and X (Theorem 1.1) and also between the fundamental groups of A and X (Theorem 2.2).

We denote the universal covering space of X by X_p^*, where $p\in X$, and the fundamental group by $\pi(X)$ [1]. All spaces are assumed to be arcwise connected, Hausdorff spaces. If α and β are continuous maps of the closed unit interval $(0, 1)$ into X, then we use the notation $\alpha\sim\beta$ (equivalent) to mean:

$\alpha(0)=\beta(0)$, $\alpha(1)=\beta(1)$, and there exists a continuous map $k(t, s)$ such that

(i) k: $(0, 1)\times(0, 1)\rightarrow X$ (into),

(ii) $k(t, 0)=\alpha(t)$, $k(t, 1)=\beta(t)$ for all $t\in(0, 1)$,

(iii) $k(0, s)=\alpha(0)=\beta(0)$, $k(1, s)=\alpha(1)=\beta(1)$ for all $s\in(0, 1)$.

A continuous map k satisfying the above conditions i, ii, and iii will be said to satisfy the E-conditions for (α, β, X).

1. LEMMA 1.1. *If A is a retract of X and $p\in A$, then A_p^* is homeomorphic to a subset of X_p^*.*

PROOF. Consider any $a^*\in A_p^*$ and any way $\alpha\in a^*$. Then $\alpha:(0, 1)\rightarrow A$. Define $\alpha':(0, 1)\rightarrow X$ by $\alpha'(t)=\alpha(t)$ for all $t\in(0, 1)$, and let x^* be the element of X_p^* such that $\alpha'\in x^*$. Defining $h(a^*)=x^*$, it is easily shown that h is single-valued.

To show that h is continuous, we consider any neighborhood $U^*(U, \alpha)$ of x^* where $h(a^*)=x^*$. Then $U^*(U, \alpha)$ consists of all g^* in X_p^* such that $g^*\supset\alpha\beta$ with $\beta(t)\subset U$ for all $t\in(0, 1)$, and where $\alpha\in x^*$ and U is any neighborhood in X of $\alpha(1)$. Consider any $\gamma\in a^*$. Then $\gamma'\in x^*$ and therefore $\gamma'\sim\alpha$. Hence there exists a continuous map $k':(0, 1)\times(0, 1)\rightarrow X$ such that k' satisfies the E-conditions for (γ', α, X). We define $k:(0, 1)\times(0, 1)\rightarrow A$ by $k(t, s)=rk'(t, s)$. It follows easily that $r\alpha(0)=\gamma(0)$, $r\alpha(1)=\gamma(1)$, and that k satisfies the E-conditions for $(r\alpha, \gamma, A)$. Hence $r\alpha\sim\gamma$ and we have $r\alpha\in a^*$. Take the neighborhood $V^*(U\cap A, r\alpha)$ of a^*. Now consider any $a_1^*\in V^*(U\cap A, r\alpha)$ and let $h(a_1^*)=x_1^*$. Then a_1^* contains a way $(r\alpha)\beta$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

where $\beta(t) \subset U \cap A$ for all $t \in (0, 1)$ and x_1^* contains the way $[(r\alpha)\beta]' = (r\alpha)'\beta'$. Since $r\alpha \in a^*$ we have $(r\alpha)' \in x^*$ and hence $(r\alpha)' \sim \alpha$. Therefore $(r\alpha)'\beta' \sim \alpha\beta'$ and $x_1^* \in U^*(U, \alpha)$. Hence h is continuous.

Using the fact that A is a retract of X, it follows easily that the map h is one-to-one.

To show h^{-1} is continuous, consider any $b^* \in h(A_p^*) \subset X_p^*$. Let $h^{-1}(b^*) = a^*$ and take any neighborhood $U^*(U \cap A, \alpha)$ of a^* where U is a neighborhood in X. Since r is continuous and $r(\alpha(1)) = \alpha(1)$, there exists a neighborhood V of $\alpha(1)$ in X such that $r(V) \subset U \cap A$. Take the neighborhood $V^*(V, \alpha') \cap h(A_p^*)$ of b^* and consider any $b_1^* \in V^*(V, \alpha') \cap h(A_p^*)$. Then $b_1^* \supset \alpha'\beta$, where $\beta(t) \subset V$ for all $t \in (0, 1)$. It follows easily that $h^{-1}(b_1^*) \supset r(\alpha'\beta) = \alpha r(\beta)$. Since $r(\beta(t)) \subset U \cap A$ for all $t \in (0, 1)$, we have that $h^{-1}(b_1^*) \in U^*(U \cap A, \alpha)$ and hence h^{-1} is continuous.

THEOREM 1.1. *If A is a retract of X and $p \in A$, then A_p^* is a retract of X_p^*.*

PROOF. Let h be the homeomorphism given by Lemma 1.1 and let $h(A_p^*) = B^* \subset X_p^*$. We shall show that B^* is a retract of X_p^*. Let r be the map that retracts X onto A, and we shall consider r to be a map of X into X. Consider any $x^* \in X_p^*$ and any way $\alpha \in x^*$. Then $r\alpha: (0, 1) \to X$, and therefore there exists an element $b^* \in X_p^*$ such that $r\alpha \in b^*$. We define the map $R: X_p^* \to B^*$ by $R(x^*) = b^*$. Since $r\alpha(t) \subset A$ for all $t \in (0, 1)$, clearly $b^* \in B^*$ and $R(X_p^*) \subset B^*$. It is easy to show that R is single-valued.

For any $b^* \in B^*$, there exists a way $\alpha' \in b^*$ such that $\alpha'(t) \subset A$ for all $t \in (0, 1)$. Hence $R(b^*) = b^*$.

To show R is continuous, consider any $x^* \in X_p^*$ and let $R(x^*) = b^*$. Consider any neighborhood $U^*(U, \alpha) \cap B^*$ of b^*. For any $\gamma \in x^*$ we have $r\gamma \in b^*$. Hence $r\gamma \sim \alpha$ and $r\gamma(1) = \alpha(1)$. Since r is continuous, for the neighborhood U of $\alpha(1)$, there exists a neighborhood V of $\gamma(1)$ such that $r(V) \subset U$. Take the neighborhood $V^*(V, \gamma)$ of x^* and consider any $x_1^* \in V^*(V, \gamma)$. Now $x_1^* \supset \gamma\beta$ where $\beta(t) \subset V$ for all $t \in (0, 1)$. There exists b_1^* such that $r(\gamma\beta) \in b_1^*$ and we have $R(x_1^*) = b_1^*$. Clearly $r(\gamma\beta) = (r\gamma)(r\beta) \sim \alpha(r\beta)$, and hence we have $\alpha(r\beta) \in b_1^*$. But since $r\beta(t) \subset U$ for all $t \in (0, 1)$, this means that $b_1^* \in U^*(U, \alpha) \cap B^*$.

2. DEFINITION. Let $C \in \pi(X)$, $\alpha \in C$. Let $x^* \in X_p^*$, $\beta \in x^*$. Define $K_C(x^*)$ to be the element of X_p^* which contains $\alpha\beta$.

It is well known that K_C is a covering homeomorphism (i.e. $K_C: X_p^* \to X_p^*$ is a homeomorphism such that $L(x^*) = LK_C(x^*)$ for all $x^* \in X_p^*$ where L is the natural map defined by $L(x^*) = \alpha(1)$, for $\alpha \in x^*$). Moreover, it is well known that the set $\{K_C\}$ of all such K_C

is simply the set of all covering homeomorphisms on X_p^* and forms a group.

For the work that follows we assume A is a retract of X. Letting h be the homeomorphism given by Lemma 1.1 and letting $h(A_p^*) = B^* \subset X_p^*$ it is easy to show the following result.

THEOREM 2.1. *Let $C \in \pi(X)$ and let $\{K_C\}$ be the collection of all covering homeomorphisms on X_p^*. Let $C_1 \in \pi(X)$ such that $C_1 \supset \alpha$ such that $\alpha(t) \in A$ for all t. Then the collection $\{K_{C_1}\}$ is the set of all covering homeomorphisms on B^*. That is, $\{K_{C_1}\}$ consists of all $K_C \in \{K_C\}$ such that $K_C: B^* \to B^*$ (onto).*

DEFINITION. A subgroup F of a group G is called a retract of G provided there exists a homomorphism h such that $h(G) = F$ and such that $h(f) = f$ for all $f \in F$.

It is easy to show that $\{K_{C_1}\}$ is a subgroup of the group $\{K_C\}$.

LEMMA 2.1. *The subgroup $\{K_{C_1}\}$ is a retract of the group of all covering homeomorphisms $\{K_C\}$ of X_p^*, and $\{K_{C_1}\}$ is isomorphic with the group of all covering homeomorphisms $\{M_C\}$ of A_p^*.*

PROOF. Define a homomorphism $H: \{K_C\} \to \{K_{C_1}\}$ as follows. Consider any $K_C \in \{K_C\}$ and any $\alpha \in C$. Then considering the retraction map $r(X) = A$ as $r: X \to X$ (into), we have $r\alpha: (0, 1) \to X$. There exists an element C_1 in $\pi(X)$ such that $r\alpha \in C_1$. Define $H(K_C) = K_{C_1}$. Clearly $K_{C_1} \in \{K_{C_1}\}$ since $r\alpha \in C_1$ and $r\alpha(t) \subset A$ for all t. It is easy to show that the homomorphism H is single-valued.

For $K_{C_1} \in \{K_{C_1}\}$, $C_1 \supset \alpha$ such that $\alpha(t) \subset A$ for all t. Hence $r\alpha(t) = \alpha(t)$ for all t and $H(K_{C_1}) = K_{C_1}$.

Consider any K_B and K_C in $\{K_C\}$ and consider any $\beta \in B$ and $\gamma \in C$. It follows easily that $K_B K_C = K_{BC}$ and hence we have $H(K_B K_C) = H(K_{BC}) = K_{(BC)_1}$ where $\beta\gamma \in BC$ and $r(\beta\gamma) \in (BC)_1$. Also $H(K_B)H(K_C) = K_{B_1}K_{C_1} = K_{B_1C_1}$ where $r\beta \in B_1$, $r\gamma \in C_1$ and hence $(r\beta)(r\gamma) \in B_1C_1$. But $r(\beta\gamma) = (r\beta)(r\gamma)$ and therefore $(BC)_1 = B_1C_1$. Hence $H(K_BK_C) = H(K_B)H(K_C)$.

Define an isomorphism $F: \{M_C\} \to \{K_{C_1}\}$ (onto) as follows. Consider any $M_C \in \{M_C\}$ and any $\alpha \in C \in \pi(A)$. Since $\alpha: (0, 1) \to A$, we define $\alpha'(t) = \alpha(t)$ for all t and have $\alpha': (0, 1) \to X$. There exists $C_1 \in \pi(X)$ such that $\alpha' \in C_1$. Define $F(M_C) = K_{C_1}$.

It is well known that the fundamental group $\pi(X)$ of X is isomorphic with the group $\{K_C\}$ of all covering homeomorphisms of X_p^*. Hence using Lemma 2.1 we have the result.

THEOREM 2.2. *If A is a retract of X, then the fundamental group $\pi(A)$ is a retract of the fundamental group $\pi(X)$.*

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DENSE IMBEDDING OF TOPOLOGICAL GROUPS

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The present article originated from the problem of determining when a continuous representation of a Lie group into another is open. Important cases of the problem have been discussed by A. Malcev[2] and the author,[3] and some of them have been extended to the case of more general groups by the author and H. Yamabe.[4]

Recently W. T. van Est obtained new results concerning the same problem on Lie groups.[5] Here I shall give an extension of his essential result to a more general case in a simpler way.

Let G be a locally compact connected group and let $A(G)$ be the group of all continuous automorphisms of G. Let $A(G)$ be topologized by the notion of uniform convergence in the wider sense.

Now let $I(G)$ be the subgroup of $A(G)$ composed of all inner automorphisms of G. We shall call G a (CA) group[6] if $I(G)$ is a closed subgroup of $A(G)$.

LEMMA.[7] *Let G be a locally compact connected and locally connected group, and H a locally compact group. If ϕ is a continuous isomorphism which maps G in an everywhere dense subgroup in H, then the following propositions hold:*

(1) *$\phi(G)$ is an invariant subgroup of H.*

(2) *Let h be an element of H. Let us consider the automorphism $\sigma_h(x)$ defined by $\sigma_h x=\phi^{-1}(h^{-1}\phi(x))h$ for $x\in G$. Then $\sigma_h(x)$ is a continuous*

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[1] The author's name will be hereafter spelled "Goto" instead of "Gotô" which has been used thus far.

[2] Malcev [4] in the bibliography.

[3] Goto [1].

[4] Goto and Yamabe [3]. See also [2].

[5] van Est [5], where he solved a prize problem (Wiskundig Genootshap Amsterdam, 1950), which had already been established in [1] and generalized in [3], independently of the author.

[6] The notion of a (CA) group is a generalization of van Est's (CA) Lie group.

[7] See [3] and [2].