A SPECIAL CONGRUENCE

L. CARLITZ

1. It is familiar that if p is a prime such that $p-1\nmid m$, $p^r|m$ then

$$B_m \equiv 0 \pmod{p^r}, \tag{1.1}$$

where B_m denotes a Bernoulli number in the even suffix notation. The writer has recently proved the companion formula ([2, Theorem 3]; see also [1])

$$B_{m(p-1)} + 1/p - 1 \equiv 0 \pmod{p^r} \qquad (p \geqq 3), \tag{1.2}$$

for $p^r|m$, $m>0$; moreover if $m=p^r h$, then

$$p^{-r}\left(B_{m(p-1)} + 1/p - 1\right) \equiv h w_p \pmod{p} \qquad (p > 3), \tag{1.3}$$

where w_p denotes Wilson's quotient $((p-1)!+1)/p$.

In this note we show that the above formulas imply

$$p + (p-1)\sum_{0<s(p-1)<m}\binom{m}{s(p-1)} \equiv 0 \pmod{p^{r+1}}, \tag{1.4}$$

where $p^r|m$ and $p\geqq 3$. More precisely if $m=p^r m_0$, we have, for $p>3$,

$$\begin{aligned} &p^{-r-1}\left\{p + (p-1)\sum_{0<s(p-1)<m}\binom{m}{s(p-1)}\right\} \\ &\equiv m_0\left\{\frac{1}{2} - \sum_{0<2s<m,\, p-1\nmid 2s}\binom{m-1}{2s-1}\frac{B_{2s}}{2s} + \delta_m\frac{w_p}{p-1}\right\} \pmod{p}, \end{aligned} \tag{1.5}$$

where $\delta_m=1$ for $p-1|m-1$, $\delta_m=0$ otherwise.

For $r=0$, (1.4) is due to Hermite. The proof below of (1.4) was suggested by Nielsen's proof [3, p. 254] of Hermite's formula.

2. **Proof of (1.4).** Using the basic recurrence for the Bernoulli numbers we may write

$$1 - \frac{1}{2}m + \sum_{0<2s<m}\binom{m}{2s}B_{2s} = 0. \tag{2.1}$$

Now let $p^r|m$. Consider first a term such that $p-1\nmid 2s$. Let $p^k|s$, so that by (1.1), $B_{2s}\equiv 0 \pmod{p^k}$. If $k\leqq r$, it follows that

$$\binom{m}{2s} = \frac{m}{2s}\binom{m-1}{2s-1} \equiv 0 \pmod{p^{r-k}} \tag{2.2}$$

Received by the editors February 2, 1953.

and consequently

$$\binom{m}{2s} B_{2s} \equiv 0 \pmod{p^r}. \tag{2.3}$$

Clearly (2.3) holds also for $k > r$. Thus (2.1) and (2.2) imply

$$1 + \sum_{0<s(p-1)<m} \binom{m}{s(p-1)} B_{s(p-1)} \equiv 0 \pmod{p^r},$$

which may be rewritten as

$$\begin{aligned} 1 + \sum_{0<s(p-1)<m} \binom{m}{s(p-1)} \left(B_{s(p-1)} + \frac{1}{p} - 1 \right) \\ \equiv \left(\frac{1}{p} - 1 \right) \sum_{0<s(p-1)<m} \binom{m}{s(p-1)} \pmod{p^r}. \end{aligned} \tag{2.4}$$

Now exactly as in proving (2.3), we may show, using (1.2), that

$$\binom{m}{s(p-1)} \left(B_{s(p-1)} + \frac{1}{p} - 1 \right) \equiv 0 \pmod{p^r}.$$

Thus (2.4) reduces to

$$1 \equiv \left(\frac{1}{p} - 1 \right) \sum_{0<s(p-1)<m} \binom{m}{s(p-1)} \pmod{p^r}. \tag{2.5}$$

It is evident that (2.5) and (1.4) are equivalent.

3. **Proof of (1.5).** We again begin with (2.1) which we now write as

$$\begin{aligned} 1 - \frac{1}{2} m + \sum_{0<2s<m,\, p-1 \nmid 2s} \binom{m}{2s} B_{2s} \\ + \sum_{0<s(p-1)<m} \binom{m}{s(p-1)} B_{s(p-1)} = 0. \end{aligned}$$

This evidently implies

$$\begin{aligned} 1 - \frac{1}{2} m + \sum_{0<2s<m,\, p-1 \nmid 2s} \binom{m}{2s} B_{2s} \\ + \sum_{0<s(p-1)<m} \binom{m}{s(p-1)} \left(B_{s(p-1)} + \frac{1}{p} - 1 \right) \\ = \sum_{0<s(p-1)<m} \binom{m}{s(p-1)} \left(\frac{1}{p} - 1 \right). \end{aligned} \tag{3.1}$$

Consider first the sum

$$(3.2)\qquad S = \sum_{0<s(p-1)<m} \binom{m}{s(p-1)}\left(B_{s(p-1)} + \frac{1}{p} - 1\right).$$

Let $p^k | s$ and put

$$s = p^k h;$$

then by (1.3) we have

$$(3.3)\qquad B_{s(p-1)} + \frac{1}{p} - 1 \equiv p^k h w_p \pmod{p^{k+1}}.$$

If $k \leqq r$ it is evident from (2.2) that (3.3) yields

$$(3.4)\qquad \binom{m}{s(p-1)}\left(B_{s(p-1)} + \frac{1}{p} - 1\right) \equiv \binom{m}{s(p-1)} p^k h w_p \pmod{p^{r+1}};$$

clearly (3.4) holds also for $k > r$. Since the right member of (3.4) is equal to

$$m\binom{m-1}{s(p-1)-1} w_p/(p-1),$$

we see that (3.2) becomes

$$(3.5)\qquad S \equiv \frac{m w_p}{p-1} \sum_{0<s(p-1)<m} \binom{m-1}{s(p-1)-1} \pmod{p^{r+1}}.$$

In the next place for the first sum in the left member of (3.1) we have

$$(3.6)\qquad \sum_{0<2s<m,\, p-1 \nmid 2s} \binom{m}{2s} B_{2s} = m \sum_{0<2s<m,\, p-1 \nmid 2s} \binom{m-1}{2s-1} \frac{B_{2s}}{2s}.$$

Substituting from (3.5) and (3.6) in (3.1) we get

$$(3.7)\qquad \begin{aligned} 1 - \left(\frac{1}{p} - 1\right) & \sum_{0<s(p-1)<m} \binom{m}{s(p-1)} \\ & \equiv \frac{1}{2} m - m \sum_{0<2s<m,\, p-1 \nmid 2s} \binom{m-1}{2s-1} \frac{B_{2s}}{2s} \\ & \quad - \frac{m w_p}{p-1} \sum_{0<s(p-1)<m} \binom{m-1}{s(p-1)-1} \pmod{p^{r+1}}. \end{aligned}$$

Now let $p^r \mid m$, $p^{r+1} \nmid m$; then (3.7) becomes

$$\frac{1}{m}\left\{1-\left(\frac{1}{p}-1\right)\sum_{0<s(p-1)<m}\binom{m}{s(p-1)}\right\}$$

$$\equiv \frac{1}{2}-\sum_{0<2s<m,\,p-1\nmid 2s}\binom{m-1}{2s-1}\frac{B_{2s}}{2s} \tag{3.8}$$

$$-\frac{w_p}{p-1}\sum_{0<s(p-1)<m}\binom{m-1}{s(p-1)-1} \pmod{p}.$$

But [3, p. 255]

$$\sum_{0<s(p-1)<m}\binom{m-1}{s(p-1)-1} \equiv \begin{cases} 0 & (p-1 \nmid m-1), \\ -1 & (p-1 \mid m-1); \end{cases} \tag{3.9}$$

indeed (3.9) is an easy consequence of the case $r=0$ of (1.4). Finally (3.7), (3.8), and (3.9) evidently imply (1.5).

References

1. L. Carlitz, *A divisibility property of the Bernoulli number*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 604–607.

2. ———, *Some congruences for the Bernoulli numbers*, Amer. J. Math. vol. 75 (1953) pp. 163–172.

3. N. Nielsen, *Sur le théorème de v. Staudt et de Th. Clausen relatif aux nombres de Bernoulli*, Annali di Matematica (3) vol. 22 (1914).

Duke University