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YALE UNIVERSITY

SOLUTION OF BERNSTEIN'S APPROXIMATION PROBLEM[1]

HARRY POLLARD

In his famous monograph on approximation theory [2], S. Bernstein initiated the study of the closure properties of sets of functions $\{u^nK(u)\}_0^\infty$ on the real line. It is supposed that $K(u)$ is continuous on $(-\infty, \infty)$ and that $u^nK(u)$ vanishes at $u=\pm\infty$ for each value of n. The problem is to decide when the set $\{u^nK(u)\}$ is fundamental in the space C_0 of functions continuous on $(-\infty, \infty)$, vanishing at $\pm\infty$, and normed by $\|f\|=\max|f(u)|$. So far no necessary and sufficient conditions have been given. A recent paper of Carleson [3] reviews most of the known results, but the paper [1] which seems to come closest to the true conditions has been overlooked.

It is the purpose of this note to give a complete solution. It applies to either real- or complex-valued functions and may be read either way.

THEOREM. *In order that $\{u^nK(u)\}_0^\infty$ be fundamental in C_0 it is necessary and sufficient that*

$$K(u) \neq 0, \qquad -\infty < u < \infty; \tag{1}$$

$$\int_{-\infty}^{\infty} \frac{\log |K(u)|}{1+u^2} du = -\infty; \tag{2}$$

and that there exists a sequence of polynomials p_n such that

$$\lim_{n\to\infty} p_n(u)K(u) = 1; \quad |p_n(u)K(u)| \leq C, \quad -\infty < u < \infty. \tag{3}$$

1. **The necessity.** The necessity of (1) is obvious and of (2) is well known [1; 3]. To prove the necessity of the remaining conditions let $0_n(u)$ denote the continuous function which is unity on $(-n, n)$, vanishes outside $(-n-1, n+1)$, and is linear in the remaining in-

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tervals. Since $\{u^n K(u)\}$ is fundamental there exists for each n a polynomial p_n such that

$$|p_n(u)K(u) - 0_n(u)| \leqq 2^{-n}.$$

Now let $n \to \infty$ and (3) follows.

2. **A lemma.** To prove the sufficiency we shall need the following result.

LEMMA. *Let $\alpha(x)$ be of bounded variation on $(-\infty, \infty)$. Then the functions*

$$F_{\pm}(x) = \alpha'(x) \pm \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\alpha(u)}{x-u}$$

exist almost everywhere when the integral is interpreted as a principal value. Moreover

$$\int_{-\infty}^{\infty} \frac{|\log |F_{\pm}(x)||}{1+x^2} dx < \infty \tag{2.1}$$

for at least one choice of the $\pm$ sign, unless α is substantially a constant.

The first part of the theorem follows from a result of Loomis [4] on Hilbert transforms. Note that (2.1) is the same for either choice of sign if α is real, so that the complication comes from the possibility that it is complex-valued.

To establish (2.1) consider the function

$$H(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\alpha(u)}{z-u}, \qquad z = x + iy.$$

$H(z)$ is analytic for $y>0$ and for $y<0$. It cannot be identically zero in both half-planes unless α is substantially a constant. Ruling out this case, we may assume $H \not\equiv 0$ in one of these half-planes, say $y>0$. We shall establish (2.1) with the $+$ sign.

Now $H = U + iV$, where $U(x, y)$ and $V(x, y)$ are defined by

$$U(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\alpha(u)}{(x-u)^2 + y^2}$$

and

$$V(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-u)d\alpha(u)}{(x-u)^2 + y^2}.$$

Since

$$\int_{-\infty}^{\infty} | U(x, y) | \, dx \leqq \int_{-\infty}^{\infty} | d\alpha(u) |$$

it follows from Schwarz's inequality that

$$\int_{-\infty}^{\infty} \frac{| U(x, y) |^{1/2}}{1 + x^2} dx \leqq C < \infty.$$

As for $V(x, y)$, an argument used by Titchmarsh [6, pp. 144–145] shows that

$$\int_{-\infty}^{\infty} \frac{| V(x, y) |^{1/2}}{1 + x^2} dx \leqq C < \infty.$$

(Titchmarsh proves this when α is an integral, but his argument is quite general.) Consequently

$$\int_{-\infty}^{\infty} \frac{| H(x + iy) |^{1/2}}{1 + x^2} dx \leqq C < \infty.$$

Map the half-plane $Iz>0$ into the unit circle $|w|<1$ by $z=i(1-w)/(1+w)$. If we write $w=re^{i\theta}$, $h(w)=H(z)$, then $d\theta=2(1+x^2)^{-1}dx$ and the preceding formula becomes

$$\int_{0}^{2\pi} | h(re^{i\theta}) |^{1/2} d\theta \leqq C < \infty, \qquad 0 \leqq r < 1.$$

A standard argument (see, for example, [5, pp. 19–20]) shows that

$$\int_{0}^{2\pi} | \log | h(re^{i\theta}) | | \, d\theta \leqq C < \infty.$$

Since h is of class $H^{1/2}$ the limit $h(e^{i\theta})=\lim_{r\to 1} h(re^{i\theta})$ exists almost everywhere. Hence by Fatou's lemma

$$\int_{0}^{2\pi} | \log | h(e^{i\theta}) | | \, d\theta \leqq C < \infty.$$

Mapping back, we get

$$\int_{-\infty}^{\infty} \frac{| \log | H(x + i0) | |}{1 + x^2} dx < \infty.$$

It remains only to identify $H(x+i0)$ with $F_+(x)$. This amounts to showing that almost everywhere

$$\lim_{y\to 0+} U(x, y) = \alpha'(x),$$

$$\lim_{y\to 0+} V(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\alpha(u)}{x-u}. \tag{2.2}$$

Each of these is well known if α is absolutely continuous [6, Chap. V]. It is therefore enough to prove them when α is singular, that is, when $\alpha'(x)=0$ almost everywhere. We shall prove only the second, (2.2), the argument for the first being similar and easier. For simplicity in printing we also write "$y\to 0$" for "$y\to 0+$".

Let x_0 be a point for which $\int_{-\infty}^{\infty} d\alpha(u)/(x-u)$ exists and for which $\alpha'(x_0)=0$. This is true for almost all x_0. By a change of variable we may assume $x_0=0$ and (2.2) becomes

$$\lim_{y\to 0}\int_{-\infty}^{\infty}\frac{u\,d\alpha(u)}{u^2+y^2} = \int_{-\infty}^{\infty}\frac{d\alpha(u)}{u}.$$

Clearly it is enough to show that

$$\lim_{y\to 0}\int_{0}^{\infty}\frac{u\,d\alpha(u)}{u^2+y^2} - \int_{y}^{\infty}\frac{d\alpha(u)}{u} = 0, \tag{2.3}$$

$$\lim_{y\to 0}\int_{-\infty}^{0}\frac{u\,d\alpha(u)}{u^2+y^2} - \int_{-\infty}^{-y}\frac{d\alpha(u)}{u} = 0.$$

We confine ourselves to (2.3).

In (2.3) replace α by $\beta=\alpha-\alpha(0)$ and integrate by parts. Since $\beta'(0)=0$, (2.3) reduces to

$$\lim_{y\to 0}\left\{-\int_{0}^{\infty}\beta(u)\frac{\partial}{\partial u}\frac{u}{u^2+y^2}\,du - \int_{y}^{\infty}\frac{\beta(u)}{u^2}\,du\right\} = 0.$$

Because $\beta(u)=o(u)$, $u\to 0$, we have

$$\int_{0}^{y}\beta(u)\frac{\partial}{\partial u}\frac{u}{u^2+y^2}\,du = o(1), \qquad y\to 0,$$

and the problem is further reduced to showing that

$$\int_{y}^{\infty}\beta(u)\left\{\frac{\partial}{\partial u}\frac{u}{u^2+y^2}+\frac{1}{u^2}\right\}du = o(1), \qquad y\to 0.$$

The last integral, after a change of variable, is

$$\int_{1}^{\infty}\frac{\beta(yu)}{yu}\,u\left\{\frac{d}{du}\frac{u}{u^2+1}+\frac{1}{u^2}\right\}du,$$

which is dominated by

$$C\int_1^\infty \left|\frac{\beta(yu)}{yu}\right|\frac{du}{u^3}.$$

Because β is bounded, $\beta(0)=0$, and $\beta'(0)=0$, the expression $\beta(yu)/yu$ approaches zero boundedly on $1\leqq u<\infty$ as $y\rightarrow 0$. Consequently the preceding expression converges to zero with y, and the proof is complete.

3. **The sufficiency.** Assume that (1), (2), (3) hold. Suppose that

$$\int_{-\infty}^{\infty} u^n K(u)d\sigma(u) = 0, \qquad n = 0, 1, \cdots, \tag{3.1}$$

where σ is of bounded variation. We must show that σ is substantially a constant.

If it is not we may form the function

$$s(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\sigma(u)}{x-u}$$

and conclude from the lemma that for some choice of the $\pm$ sign

$$\int_{-\infty}^{\infty}\frac{|\log|\sigma'(x)\pm is(x)||}{1+x^2}dx<\infty. \tag{3.2}$$

Since $K(u)\neq 0$, a similar remark applies to the function

$$g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{K(u)d\sigma(u)}{x-u}$$

and we have

$$\int_{-\infty}^{\infty}\frac{|\log|K(x)\sigma'(x)\pm ig(x)||}{1+x^2}dx<\infty. \tag{3.3}$$

It is important to know that we may choose the same sign in both (3.2) and (3.3). According to the proof of the lemma we can do this if the functions

$$S(z) = \int_{-\infty}^{\infty}\frac{d\sigma(u)}{z-u}, \qquad G(z) = \int_{-\infty}^{\infty}\frac{K(u)d\sigma(u)}{z-u}$$

have a common half-plane, $y>0$ or $y<0$, in which neither is identically zero. The identity

$$\frac{1}{z-u} = \frac{1}{z}+\frac{u}{z^2}+\cdots+\frac{u^{n-1}}{z^n}+\frac{u^n}{z^n(z-u)} \tag{3.4}$$

and (3.1) enable us to rewrite $G(z)$ as

$$G(z) = \frac{1}{z^n}\int_{-\infty}^{\infty} \frac{u^n K(u) d\sigma(u)}{z - u}.$$

Consequently for each polynomial p_n of (3) we have

$$p_n(z)G(z) = \int_{-\infty}^{\infty} \frac{p_n(u)K(u)}{z - u} d\sigma(u).$$

Since z is not real, (3) enables us to conclude that

$$\lim_{n\to\infty} p_n(z)G(z) = S(z). \tag{3.5}$$

Now $S(z)$ is not identically zero in at least one of the half-planes, say $y>0$. Hence, by (3.5), $G(z)$ cannot vanish there identically either. We may therefore assume that both (3.2) and (3.3) are valid with the $+$ sign.

In the identity (3.4) replace z by x. The resulting formula and (3.1) enable us to rewrite $g(x)$ as

$$g(x) = \frac{1}{x^n}\,\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u^n K(u) d\sigma(u)}{x - u},$$

so that

$$p_n(x)g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{p_n(u)K(u)}{x - u} d\sigma(u)$$

and

$$p_n(x)g(x) - s(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{p_n(u)K(u) - 1}{x - u} d\sigma(u).$$

By another result of Loomis [4] the measure of the set for which $|p_n(x)g(x) - s(x)| > \epsilon$ is at most

$$\frac{A}{\epsilon}\int_{-\infty}^{\infty} |p_n(u)K(u) - 1|\,|d\sigma|,$$

where A is an absolute constant. In view of (3) this approaches zero as $n\to\infty$. Hence $p_n(x)g(x)$ converges to $s(x)$ in measure, so that a subsequence converges almost everywhere to $s(x)$. By (3), $p_n(x)$ converges to $1/K(x)$. Therefore

$$g(x) = K(x)s(x)$$

for almost all x. From this identity we obtain

$$K(x) = \frac{K(x)\sigma'(x) + ig(x)}{\sigma'(x) + is(x)}.$$

Note that by (3.2) and (3.3) neither the numerator nor the denominator can vanish on a set of positive measure. Moreover by these same results

$$\int_{-\infty}^{\infty} \frac{|\log |K(x)||}{1 + x^2} dx < \infty,$$

which contradicts hypothesis (2).

Therefore σ must be substantially a constant, and the proof is complete.

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